Topic 2 of Part II
Calibration

Zhigang Zhu, City College of New York  zhu@cs.ccny.cuny.edu

Lecture Outline

- Calibration: Find the intrinsic and extrinsic parameters
  - Problem and assumptions
  - Direct parameter estimation approach
  - Projection matrix approach

- Direct Parameter Estimation Approach
  - Basic equations (from Lecture 5)
  - Homogeneous System
  - Estimating the Image center using vanishing points
  - SVD (Singular Value Decomposition)
  - Focal length, Aspect ratio, and extrinsic parameters
  - Discussion: Why not do all the parameters together?

- Projection Matrix Approach (…after-class reading)
  - Estimating the projection matrix M
  - Computing the camera parameters from M
  - Discussion

- Comparison and Summary
  - Any difference?
Problem and Assumptions

- Given one or more images of a calibration pattern, estimate the intrinsic parameters, the extrinsic parameters, or both.

- Issues: Accuracy of Calibration
  - How to design and measure the calibration pattern:
    - Distribution of control points to assure stability of solution, not coplanar
    - Construction tolerance one or two orders of magnitude smaller than desired accuracy of calibration
      - e.g., 0.01 mm tolerance versus 0.1 mm desired accuracy
  - How to extract the image correspondences:
    - Corner detection?
    - Line fitting?
  - Algorithms for camera calibration given both 3D-2D pairs

- Alternative approach: 3D from un-calibrated camera

Camera Model

- Coordinate Systems
  - Frame coordinates \((x_{im}, y_{im})\) pixels
  - Image coordinates \((x, y)\) in mm
  - Camera coordinates \((X, Y, Z)\)
  - World coordinates \((X_w, Y_w, Z_w)\)

- Camera Parameters
  - Intrinsic parameters (of the camera and the frame grabber): link the frame coordinates of an image point with its corresponding camera coordinates
  - Extrinsic parameters: define the location and orientation of the camera coordinate system with respect to the world coordinate system
3D Computer Vision
and Video Computing

Linear Version of Perspective Projection

- **World to Camera**
  - Camera: \( P = (X,Y,Z)^T \)
  - World: \( P_w = (X_w,Y_w,Z_w)^T \)
  - Transform: \( R, T \)

- **Camera to Image**
  - Camera: \( P = (X,Y,Z)^T \)
  - Image: \( p = (x,y)^T \)
  - Not linear equations

- **Image to Frame**
  - Neglecting distortion

- **World to Frame**
  - \((X_w,Y_w,Z_w)^T \rightarrow (xim, yim)^T\)
  - Effective focal lengths
    - \( f_x = f/sx, f_y = f/sy \)

**Direct Parameter Method**

- **Extrinsic Parameters**
  - \( R, 3x3 \) rotation matrix
    - Three angles \( \alpha, \beta, \gamma \)
  - \( T, 3-D \) translation vector

- **Intrinsic Parameters**
  - \( f_x, f_y \): effective focal length in pixel
    - \( \alpha = f_x/f_y = s_y/sx \), and \( f_x \)
    - \( (ox, oy) \): known Image center \( \rightarrow (x,y) \) known
    - \( k_1 \), radial distortion coefficient: *neglect it in the basic algorithm*

- **Same Denominator in the two Equations**
  - Known: \((X_w,Y_w,Z_w)\) and its \((x,y)\)
  - Unknown: \( r, p, q, T_x, T_y, f_x, f_y \)
3D Computer Vision
and Video Computing

Linear Equations

- Linear Equation of 8 unknowns $v = (v_1, \ldots, v_8)$
  - Aspect ratio: $\alpha = f_x/f_y$
  - Point pairs, $\{(X_i, Y_i, Z_i) \leftrightarrow (x_i, y_i)\}$ drop the ‘ and subscript ‘w’

$$x'(n_1X_w + n_2Y_w + n_3Z_w + T_w) = y'\alpha(n_1X_w + n_2Y_w + n_3Z_w + T_w)$$

$$x_1X_1 + x_2Y_1 + x_3Z_1 + x_4T_1 - y_1X_1 - y_2Y_1 - y_3Z_1 - y_4T_1 = 0$$

Homogeneous System

- Homogeneous System of N Linear Equations
  - Given N corresponding pairs $\{(X_i, Y_i, Z_i) \leftrightarrow (x_i, y_i)\}$, $i=1,2,\ldots,N$
  - 8 unknowns $v = (v_1, \ldots, v_8)^T$, 7 independent parameters

$$x_1X_1 + x_2Y_1 + x_3Z_1 + x_4T_1 - y_1X_1 - y_2Y_1 - y_3Z_1 - y_4T_1 = 0$$

$$\begin{bmatrix}
  x_1X_1 & x_1Y_1 & x_1Z_1 & x_1 & -y_1X_1 & -y_1Y_1 & -y_1Z_1 & -y_1 \\
  x_2X_2 & x_2Y_2 & x_2Z_2 & x_2 & -y_2X_2 & -y_2Y_2 & -y_2Z_2 & -y_2 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  x_NX_N & x_NY_N & x_NZ_N & x_N & -y_NX_N & -y_NY_N & -y_NZ_N & -y_N 
\end{bmatrix} \begin{bmatrix}
  v_1 \\
  v_2 \\
  v_3 \\
  v_4 \\
  v_5 \\
  v_6 \\
  v_7 \\
  v_8 
\end{bmatrix} = 0$$

The system has a nontrivial solution (up to a scale)
  - If $N \geq 7$ and N points are not coplanar $\Rightarrow$ Rank (A) = 7
  - Can be determined from the SVD of A
Homework #3 online, due October 25 (Monday) before class

Homogeneous System

Homogeneous System of N Linear Equations
- Given N corresponding pairs \((X_i, Y_i, Z_i) \leftrightarrow (x_i, y_i)\), \(i=1, 2, ..., N\)
- 8 unknowns \(v = (v_1, ..., v_8)^T\), 7 independent parameters

\[
x_i X_i v_1 + x_i Y_i v_2 + x_i Z_i v_3 + x_i v_4 - y_i X_i v_5 - y_i Y_i v_6 - y_i Z_i v_7 - y_i v_8 = 0
\]

\[
A v = 0
\]

\[
A = \begin{bmatrix}
x_1 X_1 & x_1 Y_1 & x_1 Z_1 & x_1 & - y_1 X_1 & - y_1 Y_1 & - y_1 Z_1 & - y_1 \\
x_2 X_2 & x_2 Y_2 & x_2 Z_2 & x_2 & - y_2 X_2 & - y_2 Y_2 & - y_2 Z_2 & - y_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_N X_N & x_N Y_N & x_N Z_N & x_N & - y_N X_N & - y_N Y_N & - y_N Z_N & - y_N
\end{bmatrix}
\]

The system has a nontrivial solution (up to a scale)
- IF \(N \geq 7\) and N points are not coplanar \(\Rightarrow \text{Rank}(A) = 7\)
- Can be determined from the SVD of A
SVD: definition

- **Singular Value Decomposition:**
  - Any mxn matrix can be written as the product of three matrices
  
  \[ A = UDV^T \]

- Singular values \( \sigma_i \) are fully determined by \( A \)
  - \( D \) is diagonal: \( d_{ij} = 0 \) if \( i \neq j \); \( d_{ii} = \sigma_i \) (\( i = 1, 2, \ldots, n \))
  - \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0 \)
  - Both \( U \) and \( V \) are not unique
  - Columns of each are mutual orthogonal vectors

SVD: properties

1. Singularity and Condition Number
   - nxn \( A \) is nonsingular IFF all singular values are nonzero
   - Condition number: degree of singularity of \( A \)
     - \( A \) is ill-conditioned if \( 1/C \) is comparable to the arithmetic precision of your machine; almost singular

2. Rank of a square matrix \( A \)
   - Rank (\( A \)) = number of nonzero singular values

3. Inverse of a square Matrix
   - If \( A \) is nonsingular \( A^{-1} = VD^{-1}U^T \)
   - In general, the pseudo-inverse of \( A \) \( A^+ = VD_0^{-1}U^T \)

4. Eigenvalues and Eigenvectors (questions)
   - Eigenvalues of both \( A^TA \) and \( AA^T \) are \( \sigma_i^2 \) (\( \sigma_i > 0 \))
   - The columns of \( U \) are the eigenvectors of \( AA^T \) (nxm)
   - The columns of \( V \) are the eigenvectors of \( A^TA \) (nxn)
### SVD: Application 1

**Least Square**
- Solve a system of \( m \) equations for \( n \) unknowns \( x(m \geq n) \)
- \( A \) is a \( m \times n \) matrix of the coefficients
- \( b \neq 0 \) is the \( m \)-D vector of the data
- Solution:
  
  \[
  A^T A x = A^T b
  \]

  \[
  x = (A^T A)^+ A^T b
  \]

- **How to solve:** compute the pseudo-inverse of \( A^T A \) by SVD
  - \((A^T A)^+\) is more likely to coincide with \((A^T A)^{-1}\) given \( m > n \)
  - Always a good idea to look at the condition number of \( A^T A \)

### SVD: Application 2

**Homogeneous System**
- \( m \) equations for \( n \) unknowns \( x(m \geq n-1) \)
- Rank \( (A) = n-1 \) (by looking at the SVD of \( A \))
- A non-trivial solution (up to an arbitrary scale) by SVD:
  - Simply proportional to the eigenvector corresponding to the only zero eigenvalue of \( A^T A \) (\( n \times n \) matrix)

  \[
  A^T A v_i = \sigma_i^2 v_i
  \]

- **Note:**
  - All the other eigenvalues are positive because \( \text{Rank} (A) = n-1 \)
  - For a proof, see Textbook p. 324-325
  - In practice, the eigenvector (i.e. \( v_n \)) corresponding to the minimum eigenvalue of \( A^T A \), i.e. \( \sigma_n^2 \)
**Problem Statements**
- Numerical estimate of a matrix $A$ whose entries are not independent
- Errors introduced by noise alter the estimate to $\hat{A}$

**Enforcing Constraints by SVD**
- Take orthogonal matrix $A$ as an example
- Find the closest matrix to $\hat{A}$, which satisfies the constraints exactly
  - SVD of $\hat{A}$: $\hat{A} = UDV^T$
  - Observation: $D = I$ (all the singular values are 1) if $A$ is orthogonal
  - Solution: changing the singular values to those expected

$$A = UIV^T$$

---

**Homogeneous System**

- Homogeneous System of N Linear Equations
  - Given $N$ corresponding pairs $\{(X_i, Y_i, Z_i) \leftrightarrow (x_i, y_i)\}$, $i=1,2,\ldots,N$
  - 8 unknowns $v = (v_1, \ldots, v_8)^T$, 7 independent parameters
- The system has a nontrivial solution (up to a scale)
  - IF $N \geq 7$ and $N$ points are not coplanar $\Rightarrow$ Rank $(A) = 7$
  - Can be determined from the SVD of $A$
  - Rows of $V^T$: eigenvectors $\{e_i\}$ of $A^TA$
  - Solution: the $8^{th}$ row $e_8$ corresponding to the only zero singular value $\lambda_8 = 0$

$$A = UDV^T$$

$$\overline{v} = c e_8$$

- Practical Consideration
  - The errors in localizing image and world points may make the rank of $A$ to be maximum (8)
  - In this case select the eigenvector corresponding to the smallest eigenvalue.
Equations for scale factor $\gamma$ and aspect ratio $\alpha$

$$\mathbf{V} = \gamma \left( r_{21}, r_{22}, r_{23}, T_y, \alpha r_{11}, \alpha r_{12}, \alpha r_{13}, \alpha T_x \right)$$

Knowledge: $\mathbf{R}$ is an orthogonal matrix

$$\mathbf{R}^T_i \mathbf{R}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Second row ($i=j=2$):

$$r_{21}^2 + r_{22}^2 + r_{23}^2 = 1$$

First row ($i=j=1$)

$$r_{11}^2 + r_{12}^2 + r_{13}^2 = 1$$

Equations for first 2 rows of $\mathbf{R}$ and $\mathbf{T}$ given $\alpha$ and $|\gamma|$

$$\mathbf{V} = s |\gamma| \left( r_{21}, r_{22}, r_{23}, T_y, \alpha r_{11}, \alpha r_{12}, \alpha r_{13}, \alpha T_x \right)$$

First 2 rows of $\mathbf{R}$ and $\mathbf{T}$ can be found up to a common sign $s$ (+ or -)

$$s \mathbf{R}^T_1, s \mathbf{R}^T_2, s \mathbf{T}_X, s \mathbf{T}_Y$$

The third row of the rotation matrix by vector product

$$\mathbf{R}^T_3 = \mathbf{R}^T_1 \times \mathbf{R}^T_2 = s \mathbf{R}^T_1 \times s \mathbf{R}^T_2$$

Remaining Questions:
- How to find the sign $s$?
- Is $\mathbf{R}$ orthogonal?
- How to find $T_z$ and $f_x, f_y$?
### Find the sign \(s\)

- **Facts:**
  - \(f_x > 0\)
  - \(Z_c > 0\)
  - \(x\) known
  - \(X_w, Y_w, Z_w\) known
- **Solution**
  - Check the sign of \(X_c\)
  - Should be opposite to \(x\)

\[
x = -f_x \frac{X_c}{Z_c} = -f_x \left( \frac{\eta_1 X_w + \eta_2 Y_w + \eta_3 Z_w + T_x}{r_{31} X_w + r_{32} Y_w + r_{33} Z_w + T_z} \right)
\]

\[
y = -f_y \frac{Y_c}{Z_c} = -f_y \left( \frac{r_{21} X_w + r_{22} Y_w + r_{23} Z_w + T_y}{r_{31} X_w + r_{32} Y_w + r_{33} Z_w + T_z} \right)
\]

### Rotation \(R\): Orthogonality

- **Question:**
  - First 2 rows of \(R\) are calculated without using the mutual orthogonal constraint

\[
\hat{R}^T \hat{R} = I? 
\]

- **Solution:**
  - Use SVD of estimate \(R\)

\[
\hat{R} = UDV^T \quad \Rightarrow \quad R = UIV^T
\]

Replace the diagonal matrix \(D\) with the 3x3 identity matrix.
Find $T_z$, $F_x$ and $F_y$

Solution
- Solve the system of $N$ linear equations with two unknowns $T_x$, $F_x$

$$x = -F_x \left( n_1X_w + n_2Y_w + n_3Z_w + T_x \right)$$

$$r_3X_w + r_2Y_w + r_1Z_w + T_z$$

- Least Square method

$$\begin{bmatrix} T_z \\ F_x \end{bmatrix} = \left( A^T A \right)^{-1} A^T b$$

- SVD method to find inverse

Direct parameter Calibration Summary

Algorithm (p130-131)
1. Measure $N$ 3D coordinates $(X_i, Y_i, Z_i)$
2. Locate their corresponding image points $(x_i, y_i)$ - Edge, Corner, Hough
3. Build matrix $A$ of a homogeneous system $Av = 0$
4. Compute SVD of $A$, solution $v$
5. Determine aspect ratio $\alpha$ and scale $|\gamma|$
6. Recover the first two rows of $R$ and the first two components of $T$ up to a sign
7. Determine sign $s$ of $\gamma$ by checking the projection equation
8. Compute the 3rd row of $R$ by vector product, and enforce orthogonality constraint by SVD
9. Solve $T_z$ and $F_x$ using Least Square and SVD, then $F_y = F_x / \alpha$
Homework #3 online, due October 25 before class

Questions

- Can we select an arbitrary image center for solving other parameters?

- How to find the image center \((ox, oy)\)?

- How about to include the radial distortion?

- Why not solve all the parameters once?

  - How many unknown with \(ox, oy\)? --- 20 ??? – projection matrix method

\[
x = x_{im} - o_x = -f_x \left( \eta_1 X_w + \eta_2 Y_w + \eta_3 Z_w + T_x \right)
= r_{31} X_w + r_{32} Y_w + r_{33} Z_w + T_z
\]

\[
y = y_{im} - o_y = -f_y \left( \eta_1 X_w + \eta_2 Y_w + \eta_3 Z_w + T_y \right)
= r_{31} X_w + r_{32} Y_w + r_{33} Z_w + T_z
\]
Vanishing points:

- Due to perspective, all parallel lines in 3D space appear to meet in a point on the image - the vanishing point, which is the common intersection of all the image lines.
Vanishing points:

- Due to perspective, all parallel lines in 3D space appear to meet in a point on the image - the vanishing point, which is the common intersection of all the image lines.

Important property:

- Vector OV (from the center of projection to the vanishing point) is parallel to the parallel lines.
Orthocenter Theorem:
- Input: three mutually orthogonal sets of parallel lines in an image
- T: a triangle on the image plane defined by the three vanishing points
- Image center = orthocenter of triangle T
- Orthocenter of a triangle is the common intersection of the three altitudes
Orthocenter Theorem:
- Input: three mutually orthogonal sets of parallel lines in an image
- T: a triangle on the image plane defined by the three vanishing points

- Image center = orthocenter of triangle T
- Orthocenter of a triangle is the common intersection of the three altitudes

Assumptions:
- Known aspect ratio
- Without lens distortions

Questions:
- Can we solve both aspect ratio and the image center?
- How about with lens distortions?
Direct parameter Calibration Summary

Algorithm (p130-131)

1. Estimate image center (Xi, Yi, Zi)
2. Measure N 3D coordinates (Xi, Yi, Zi)
3. Build matrix A of a homogeneous system Av = 0
4. Compute SVD of A, solution v
5. Determine aspect ratio α and scale |γ|
6. Recover the first two rows of R and the first two components of T up to a sign
7. Determine sign s of γ by checking the projection equation
8. Compute the 3rd row of R by vector product, and enforce orthogonality constraint by SVD
9. Solve Tz and fx using Least Square and SVD, then fy = fx / α

Remaining Issues and Possible Solution

Original assumptions:
- Without lens distortions
- Known aspect ratio when estimating image center
- Known image center when estimating others including aspect ratio

New Assumptions
- Without lens distortion
- Aspect ratio is approximately 1, or α = fx/fy = 4:3; image center about (M/2, N/2) given a MxN image

Solution (?)
1. Using α = 1 to find image center (ox, oy)
2. Using the estimated center to find others including α
3. Refine image center using new α; if change still significant, go to step 2; otherwise stop

Projection Matrix Approach
- Homework #3 online, due October 25 before class

### Linear Matrix Equation of perspective projection

- **Projective Space**
  - Add fourth coordinate
  - Define \((u,v,w)\) such that \(u/w = x_{im}, v/w = y_{im}\)
- **3x4 Matrix \(E_{\text{ext}}\)**
  - Only extrinsic parameters
  - World to camera
- **3x3 Matrix \(E_{\text{int}}\)**
  - Only intrinsic parameters
  - Camera to frame

- **Simple Matrix Product!**
  - Projective Matrix \(M = M_{\text{int}}M_{\text{ext}}\)
  - Linear Transform from projective space to projective plane
  - \(M\) defined up to a scale factor – 11 independent entries
3D Computer Vision

and Video Computing

Projection Matrix $M$

- **World – Frame Transform**
  - Drop “im” and “w”
  - N pairs $(x_i, y_i) \leftrightarrow (X_i, Y_i, Z_i)$
  - Linear equations of $m$

$$\begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} = M \begin{bmatrix} X_i \\ Y_i \\ Z_i \\ 1 \end{bmatrix}$$

$Am = 0$

- **3x4 Projection Matrix $M$**
  - Both intrinsic (4) and extrinsic (6) – 10 parameters

$$M = \begin{bmatrix} -f_x r_1 + o_x r_3 & -f_x r_2 + o_y r_3 & -f_x n_3 + o_x n_3 & -f_x T_x + o_x T_z \\ -f_y r_2 + o_y r_3 & -f_y r_2 + o_y r_3 & -f_y n_3 + o_y n_3 & -f_y T_y + o_y T_z \\ r_3 1 & r_3 2 & r_3 3 & T_z \end{bmatrix}$$

**Step 1: Estimation of projection matrix**

- **World – Frame Transform**
  - Drop “im” and “w”
  - N pairs $(x_i, y_i) \leftrightarrow (X_i, Y_i, Z_i)$

- **Linear equations of $m$**
  - $2N$ equations, 11 independent variables
  - $N \geq 6$ , SVD => $m$ up to a unknown scale

$$\begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & 0 & 0 & 0 & 0 & -x_1X_1 & -x_1Y_1 & -x_1Z_1 & -x_1 \\ 0 & 0 & 0 & X_1 & Y_1 & Z_1 & 1 & -y_1X_1 & -y_1Y_1 & -y_1Z_1 & -y_1 \end{bmatrix}$$

$$m = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{21} & m_{22} & m_{23} & m_{24} & m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix}^T$$
Step 2: Computing camera parameters

- 3x4 Projection Matrix $\hat{M}$
  - Both intrinsic and extrinsic

$$\hat{M} = \begin{bmatrix} q_1 & q_{41} \\ q_2 & q_{42} \\ q_3 & q_{43} \end{bmatrix}$$

- From $\hat{M}$ to parameters (p134-135)
  - Find scale $|\gamma|$ by using unit vector $R_3^T$
  - Determine $T_z$ and sign of $\gamma$ from $m_{34}$ (i.e. $q_{43}$)
  - Obtain $R_3^T$
  - Find $(O_x, O_y)$ by dot products of Rows $q_1, q_3, q_2, q_3$, using the orthogonal constraints of $R$
  - Determine $f_x$ and $f_y$ from $q_1$ and $q_2$ (Eq. 6.19) Wrong??)
  - All the rests: $R_1^T, R_2^T, T_x, T_y$
  - Enforce orthogonality on $R$?

Comparisons

- Direct parameter method and Projection Matrix method

- Properties in Common:
  - Linear system first, Parameter decomposition second
  - Results should be exactly the same

- Differences
  - Number of variables in homogeneous systems
    - Matrix method: All parameters at once, 2N Equations of 12 variables
    - Direct method in three steps: N Equations of 8 variables, N equations of 2 Variables, Image Center – maybe more stable
  - Assumptions
    - Matrix method: simpler, and more general; sometime projection matrix is sufficient so no need for parameter decomposition
    - Direct method: Assume known image center in the first two steps, and known aspect ratio in estimating image center
Guidelines for Calibration

- Pick up a well-known technique or a few
- Design and construct calibration patterns (with known 3D)
- Make sure what parameters you want to find for your camera
- Run algorithms on ideal simulated data
  - You can either use the data of the real calibration pattern or using computer generated data
  - Define a virtual camera with known intrinsic and extrinsic parameters
  - Generate 2D points from the 3D data using the virtual camera
  - Run algorithms on the 2D-3D data set
- Add noises in the simulated data to test the robustness
- Run algorithms on the real data (images of calibration target)
- If successful, you are all set
- Otherwise:
  - Check how you select the distribution of control points
  - Check the accuracy in 3D and 2D localization
  - Check the robustness of your algorithms again
  - Develop your own algorithms

3D reconstruction using two cameras

Stereo Vision

& project discussions

- Homework #3 online, due October 25 before class