Minimum Connected Camera Network

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1 Minimum Connected Camera Network

Given a polygon P of n vertices, we would like to place camera sensors inside P such that the cameras together guard the polygon P (i.e., each point of P is visible to at least one camera), and the cameras form a connected network if each node has a communication range of radius r. The problem of *Minimum Connected Camera Network* asks for such a network of a minimum number of cameras.

When $r = \infty$, the problem becomes the classical Art Gallery Problem. For any fixed constant r, one can shrink a polygon P such that the diameter of P is below P. Thus the problem of finding the optimal solution for any fixed r is NP-hard.

When $r = 0$, we call this problem the *Minimum Connected Guarding Network*. It is known that this problem is NP-hard in a non-simple polygon by reduction from rectilinear Minimum Steiner tree problem. And that a watchman tour is a 2-approximation.

There are two versions of this problem, depending on the definition of communication range. In the version OPT-IN, two camera nodes u, v are connected by a communication link if and only if $|uv| \leq r$ and the line segment uv is entirely inside P. In the version OPT-OUT, the line segment uv may be partially outside of P. Obviously OPT-IN $\geq$ OPT-OUT, since OPT-IN is more restrictive. There is an example (the example of a pair of "pants") which shows that OPT-IN can be a lot greater than OPT-OUT.

We considered possible lower bounds for OPT-IN. Obviously the art gallery solution is a lower bound for any r. Another lower bound is the optimal watchman tour length L divided by $2r$.

Lemma 1.1. *Suppose the optimal watchman tour has length L. Then*

$$\textit{OPT-IN} \geq \frac{L}{2r} + 1.$$

Proof: Take the optimal solution for OPT-IN which is a connected network of k nodes. We can find a spanning tree of this network which has $k - 1$ edges each with length at most r. Now this tree is completely inside P. We double the tree edges and this becomes a tour of length at most $2r(k - 1)$ and is a watchman tour. Thus the length is at least as big as L, the optimal watchman tour length. $\square$

As an upper bound, an easy one is $\mathcal{O}(\lfloor n/3 \rfloor + \mathrm{Peri}(P)/(2r))$ by simply walking along the polygon perimeter to connect an art gallery solution of $\lfloor n/3 \rfloor$ guards. In the worst case, we may need $\Omega(\lfloor n/3 \rfloor + \mathrm{Peri}(P)/(2r))$ sensors for OPT-IN where $\mathrm{Peri}(P)$ is the perimeter of P.

An approximation algorithm for any specific polygon P is still unknown. The attempt of placing guards only on a watchman tour solution fails since we may need to put a much larger number of guards to achieve coverage. The tricky part seems to be a good way of combining the watchman tour solution to the art gallery solution.

2 Minimum Guard Network

Given a polygon P of n vertices, we would like to place camera sensors inside P such that the cameras together guard the polygon P (i.e., each point of P is visible to at least one camera), and the visibility graph of the guards is connected.

For this problem we show that $\lceil n/2 - 1 \rceil$ is always sufficient and sometimes necessary in a simple polygon.

For the upper bound, we take a triangulation of the polygon P and consider the dual graph which is a tree of $n-2$ nodes with degree at most 3. We prove the claim by induction. Each time we take out two triangles $\triangle_1, \triangle_2$ at a time, use one guard g to cover them and g stays on the shared boundary of $\triangle_1 \cup \triangle_2$ with the rest of P (denoted by P'). Now we use induction hypothesis to find a set of $\lceil n/2 - 2 \rceil$ guards G' to cover P'. G' form a connected network by visibility. Since all points of P' are guarded. The guard g on the boundary of P' must be visible to at least one guard in G'. Therefore all guards $G' \cup \{g\}$ form a connected network.

What is left to prove is the selection of $\triangle_1, \triangle_2$ and g of the above property. There are two cases. Take a leaf node $\triangle_1$ and consider its parent triangle. If the parent triangle has degree 2 in the dual graph, we denote it as $\triangle_2$ as required. We take the shared vertex of $\triangle_1, \triangle_2$ which also stays on the boundary of $\triangle_1 \cup \triangle_2$ with P'.

If the parent triangle of $\triangle_1$ has degree 3, we denote the parent node as $\triangle_3$. $\triangle_3$ has three edges, two of them are shared with $\triangle_1$ and another leaf triangle $\triangle_2$. In this case we will take the common vertices shared by all three triangles as g. This will be as required.

We also knows that for any given polygon P we can find a $2k$ approximation if one can find a k approximation to the art gallery problem. This is because the visibility region of an art gallery solution form a connected network. Thus placing additional guards at certain intersections of these visibility regions can form a connected set of guards.

Related work. Val Pinciu worked on the problem of *guarded guards* in a polygon, in which a minimum set of guards are selected such that the polygon P is guarded and each guard must see at least one other guard. However, the visibilty network may not be connected. Pinciu showed that $3n/7$ guards are sufficient and sometimes necessary.