Curve Mowing*

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At a June 2015 computational geometry workshop, Jie Gao proposed the following question: given a circular mower and a path, can we tell whether that path is optimal for mowing its induced mowing region? More formally,

Question 1. *Let $C(t)$ be a continuous, closed curve in the plane parameterized by $t \in [0,1]$, and let $r > 0$. Let $P = C(t) + B(0,r)$ denote the Minkowski sum of $C(t)$ and a radius-r ball centered at the origin. For which $C(t)$ does there exist $C'(t)$ such that*

1. $C(t) + B(0,r) \subseteq C'(t) + B(0,r)$ *– (Covers original region),*
2. $L(C') < L(C)$ *– (Shorter path length)?*

If no such curves C' exist then we say that C is optimal. We are interested in characterizing when a curve is optimal, and finding alorithms for deciding and witnessing (non-)optimality. We note that this is closely related to the mowing problem studied in Arkin et al. [AFM00] in which an area to mow is given as input. We are also interested in finding local rules to deform a non-optimal path towards optimality. For computational purposes we are especially interested in the case where $C(t)$ is polygonal.

Relevant characteristics of the curve include convexity, simplicity, and whether any crossings are transversal. Relevant parameters include the mower radius r, the path curvature, and the inner diameter d_C of C defined to be

$$d_C = \inf\{r > 0 : \exists t, t' \in [0,1] \;\; C(t') \in B(C(t), r) \wedge B(C(t), r) \cap C \text{ is not path connected}\}.$$

In the degenerate case when a curve is homeomorphic to a line segment we set $d_C = 0$.

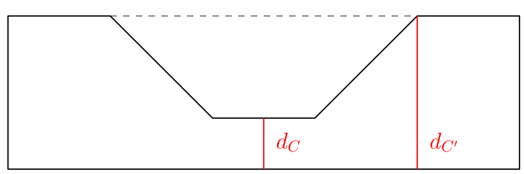

Figure 1: The inner diameters of a non-convex curve C and its convex hull C' (shown with a dashed line).

1 Sketch of results

Claim 1. *Let $C(t)$ be a simple convex curve. Then $C(t)$ is optimal for any $r > 0$.*

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†Anika Rounds, Joe Mitchell, Kiril Solovey and other workshop participants also contributed and made helpful comments.

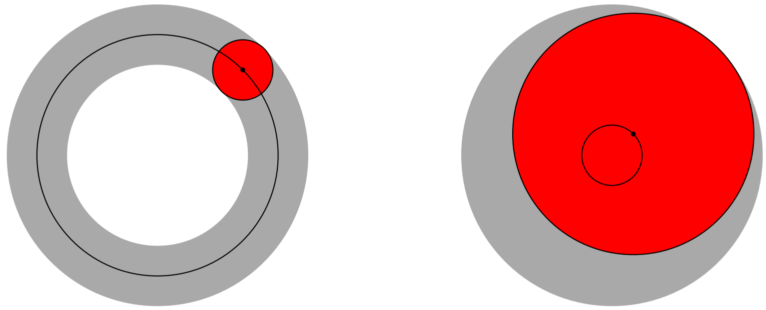

Figure 2: Mowing paths around circles in the regimes where $r < d_C$ (left, no overlap, mows an annulus) and $r > d_C$ (right, overlap, mows a disk). By Claim 1 both paths are optimal. The black dot shows $C(0)$, the inner black curve intersecting $C(0)$ shows all of $C(t)$, the red region shows the footprint of the mower at $C(0)$, and the gray region shows the total region mowed region $C(t) + B(0, r)$.

Claim 2. *Let $C(t)$ be a simple curve. Then $C(t)$ is optimal for any $r \leq \frac{1}{2} d_C$.*

Claim 3. *Let $C(t)$ be a curve whose convex hull $C'(t)$ has inner diameter $d_{C'}$. Then $C'(t)$ is optimal for any $r \geq \frac{1}{2} d_{C'}$.*

Note that Claim 3 implies that if $C \neq C'$ then C was non-optimal. Intuitively Claim 3 says that given a large enough mower we will mow the entire interior of C even if we mow along the shorter convex hull C' instead.

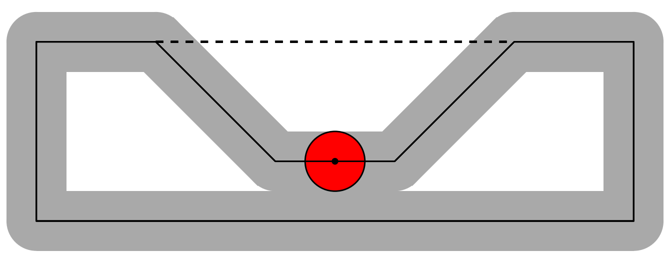

Figure 3: A non-convex curve C with $r = \frac{1}{2} d_C$. This is still optimal by Claim 2. However if $r \geq \frac{1}{2} d_{C'}$, where C' is the convex hull of C, then deforming the non-convex part of C to the convex hull C' (shown with a dashed line) would be optimal by Claim 3.

2 Things to address

- Relationship to Arkin et al. results including hardness. Relationship to other existing work.
- Local optimization/non-optimality witnesses. When we can shorten curve locally via translations and rotations.
- Algorithm for checking optimality of a curve locally.
- Non-simple curves. Uncrossings.

References

[AFM00] Esther M. Arkin, Sándor P. Fekete, and Joseph S. B. Mitchell. Approximation algorithms for lawn mowing and milling. *Comput. Geom.*, 17(1-2):25–50, 2000.