Guarding Orthogonal Polygons with h Holes: A Bound Independent of h

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Abstract

Hoffmann and Kriegel showed that an orthogonal gallery with n vertices and an unspecified number of holes can be protected by at most $n/3$ vertex guards. We improve this bound to $(17n-8)/52$.

1 Introduction

The original art gallery problem, posed by Klee in 1973, asks to find the minimum number of guards sufficient to cover any polygon with n vertices. The first solution to this problem was given by Chvátal [2], who proved that $\lfloor n/3 \rfloor$ guards are sometimes necessary, and always sufficient to cover a polygon with n vertices. Later Fisk [4] provided a shorter proof of Chvátal's theorem using an elegant graph coloring argument. Klee's art gallery problem has since grown into a significant area of study. Numerous *art gallery problems* have been proposed and studied with different restrictions placed on the shape of the galleries or the powers of the guards. (See the monograph by O'Rourke [12], and the surveys by Shermer [13] and Urrutia [14].)

Throughout this paper P denotes an orthogonal polygon with n vertices and h holes. We study *unholey* bounds for the minimum number of vertex guards needed to protect P—bounds that are independent of h and involve n only. If the guards can be posted away from the vertices of P, then $\lfloor n/4 \rfloor$ guards are sometimes necessary and always sufficient [6]. For vertex guards the best lower and upper bounds known are in the following result.

Theorem 1 *For an orthogonal art gallery with n vertices and an unspecified number of holes*

(a) $\lfloor 2n/7 \rfloor$ *vertex guards are sometimes necessary* [6]*;*
(b) $\lfloor n/3 \rfloor$ *vertex guards are sufficient* [7]*.*

The gap between $2/7 \doteq 0.2857$ and $1/3 \doteq 0.3333$ in Theorem 1 has been difficult to close; Hoffmann conjectures that his lower bound $\lfloor 2n/7 \rfloor$ in (a) is sharp. The upper bound $\lfloor n/3 \rfloor$ in (b) is a consequence of the sophisticated work by Hoffmann and Kriegel [7] on triangulations of plane bipartite graphs. We introduce same-sign diagonal graphs and use them to establish the upper bound $\lfloor (17n-8)/52 \rfloor$, which improves the bound of Hoffmann and Kriegel ($17/52 \doteq 0.3269$).

2 Diagonal Graphs in Orthogonal Polygons

The following theorem of Kahn, Klawe, and Kleitman [8], is fundamental to guarding problems for orthogonal polygons.

Theorem 2 *An orthogonal polygon can be partitioned into convex quadrilaterals by inserting suitable non-crossing diagonals.*

The configuration resulting from applying Theorem 2 to a polygon with holes defines a *quadrangulation graph* $\mathcal{Q} = (V, E)$, which is bipartite with a bipartition $V = V^+ \cup V^-$ into positive and negative vertices. The bipartition is unique when $h = 0$, but it can depend on the particular convex quadrangulation selected when $h \neq 0$. When we refer to positive and negative vertices in an orthogonal polygon with holes, it is always with respect to a particular convex quadrangulation.

Fix a convex quadrangulation of an orthogonal polygon P with n vertices and h holes. We define the *positive diagonal graph* $\mathcal{G}^+ = (V^+, E^+)$ on the positive vertex set V^+ by joining two positive vertices by an edge provided they occur in a common quadrilateral. Each edge of $\mathcal{G}^+$ is a diagonal of a quadrilateral. The *negative diagonal graph* $\mathcal{G}^- = (V^-, E^-)$ is defined similarly.

The following results give basic properties of the same-sign diagonal graphs that arise from an orthogonal polygon with holes.

Lemma 3 *In a convex quadrangulation of an orthogonal polygon with n vertices and h holes the same-sign diagonal graphs are connected plane graphs with $n/2$ vertices, $(n/2) + h - 1$ edges, and $h + 1$ faces. Each of the h bounded faces contains exactly one hole.*

Lemma 4 *There are no 3-cycles in the diagonal graphs $\mathcal{G}^+$ and $\mathcal{G}^-$ of a convex quadrangulation of an orthogonal polygon with holes.*

3 Vertex Covers

A *vertex cover* for a graph $\mathcal{G}$ is a vertex subset S such that every edge of $\mathcal{G}$ is incident with at least one vertex in S. The importance of vertex covers for guarding problems is made clear by the following observation. In

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a convex quadrangulation of an orthogonal polygon the diagonal graphs $\mathcal{G}^+$ and $\mathcal{G}^-$ both have one edge in each convex quadrilateral. Therefore a vertex cover for $\mathcal{G}^+$ or $\mathcal{G}^-$ is a guard set of the polygon.

A set of vertices in a graph $\mathcal{G}$ is *stable* provided no two vertices in the set are adjacent. The complement of a stable set is a vertex cover for $\mathcal{G}$. Thus a lower bound for the cardinality of a stable set translates to an upper bound for the cardinality of a vertex cover. A result of Kreher and Radziszowski [9] (also see [5]) on stable sets takes the following form for vertex covers.

Proposition 5 *A graph $\mathcal{G} = (V, E)$ with no 3-cycles has a vertex cover of cardinality at most*

$$\left\lfloor \frac{7|V| + |E|}{13} \right\rfloor.$$

4 Main Result

With Proposition 5 we readily deduce the following same-sign bounds for guarding orthogonal polygons with holes.

Theorem 6 *An orthogonal polygon with n vertices and h holes has a positive guard set and a negative guard set, each of cardinality at most*

$$\left\lfloor \frac{4n + h - 1}{13} \right\rfloor.$$

In particular, the polygon has a positive guard set and a negative guard set of cardinality at most

$$\left\lfloor \frac{17n - 8}{52} \right\rfloor.$$

5 Remarks

Same-sign diagonal graphs were introduced by the authors in [10], where we generalize and unify the known results about guarding orthogonal polygons and obtain simpler proofs. We use the same-sign diagonal graphs of a convex quadrangulation to deduce upper bounds for the number of guards from elementary results about vertex covers for graphs. Our method often guarantees two disjoint vertex guard sets of relatively small cardinality. For instance, we show that an orthogonal art gallery on n vertices has two disjoint guard sets of cardinality at most $n/4$ and two disjoint guarded guard sets of cardinality at most $n/3$. We also give new proofs of Aggarwal's one-hole theorem and the orthogonal fortress theorem.

Our method also yields new theorems such as Theorem 6. Also, an orthogonal gallery with n vertices and h holes can be protected by $\lfloor (n + 2h)/3 \rfloor$ guarded guards, which is best possible when $n \geq 16h$. Moreover, for orthogonal fortresses with n vertices, $\lfloor (n + 6)/3 \rfloor$ guarded guards are always sufficient and sometimes necessary.

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