Vertex-Transitive Polyhedra in 3-Space

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Abstract

In addition to regular and chiral polyhedra, which have been extensively studied, the vertex-transitive polyhedra of higher genus also present an attractive and worthwhile challenge. While the definition is combinatorial, the problem at hand is the realization in Euclidean 3-space as a highly symmetric, non-self-intersecting polyhedron in the more classical sense (with flat, non-self-intersecting faces). This abstract provides an overview of the topic as well as work in progress.

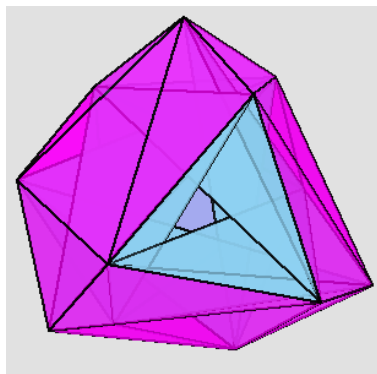

Figure 1: The only higher genus polyhedron with vertex-transitivity under the tetrahedral group of rotations.

1 Introduction

A *polyhedron* P is a closed, compact, orientable surface embedded in $\mathbb{E}^3$, tiled by finitely many plane simple polygons (*faces*) in a face-to-face manner. It can be considered a *polyhedral realization*, that is, an embedding into 3-space, of an underlying *polyhedral map* M on an abstract surface S. Along with the polyhedron P, we have a combinatorial symmetry group Γ (acting on M) and a geometric symmetry group G (isometries of $\mathbb{E}^3$ fixing P). P is *(geometrically) vertex-transitive* if G acts transitively on the vertices of P. All combinatorial types of vertex-transitive polyhedra of genus zero are known; they comprise the Platonic solids, the Archimedean solids, and the infinite families of prisms and antiprisms. However, only a few examples of higher genus are known, and the completeness of the list has never been established.

2 Known Results

The first systematic investigation of vertex-transitive polyhedra as objects in 3-space appears to have been done by Grünbaum and Shep-

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hard [2]. They established that vertex-transitive polyhedra can have positive genus, specifically $g = 1, 3, 5, 7, 11, 19$. In addition to an infinite family of tori based on antiprisms, they gave five examples of higher genus based on snub versions of the Platonic solids (with suitable tunnels).

A sixth example for $g \geq 2$ is the combinatorially regular Grünbaum polyhedron of genus three, which was discovered and rediscovered several times. It is discussed in detail in the recent survey by Gévay, Schulte, and Wills [1], who also found a seventh example, of genus 11.

3 Outline of the Approach

It can be verified (c.f. [2]) that all vertices of a vertex-transitive polyhedron have to lie on a sphere, and that all faces must be convex polygons. More recently, it has been established [1] that the only possible symmetry groups of vertex-transitive polyhedra of genus $g \geq 2$ are the rotation groups of the Platonic solids. Consequently, there are finitely many vertex-transitive polyhedra in total, not just in each genus. Tucker [3] investigated how the genus of a surface limits the possible geometric symmetries.

The problem can now be approached one symmetry group at a time, for the tetrahedral, octahedral, and icosahedral group of rotations. As a first step, polyhedral maps with vertex-transitive action of the chosen group *and* appropriate $g \geq 2$ have to be enumerated. This can be done computationally. Not all of these will be realizable. Next, it is necessary to find obstructions to realizability, which is a rather intricate process. Finally, if the previous steps prove inconclusive, it may be worthwhile to attempt construction.

For the case of tetrahedral symmetry, the list of candidate maps is short. All but two can be almost immediately excluded for realizability. Then, an additional obstruction can be found for one of the remaining two maps (proof omitted here). The other map gives rise to one of the polyhedra, of genus $g = 3$, already known to Grünbaum and Shephard [2]. A realization of the polyhedron is depicted in Figure 1.

Theorem 1. *There is exactly one combinatorial type of polyhedron of $g \geq 2$ which can be embedded into 3-space with vertex-transitive tetrahedral symmetry.*

Research into the octahedral and icosahedral case is ongoing, and is done by the author as part of her dissertation. Material in this abstract has been presented previously in the Special Session on Discrete Geometry of Polytopes in the 2013 AMS Spring Eastern Sectional Meeting.

References

[1] G. Gévay, E. Schulte, and J. M. Wills. The Regular Grünbaum Polyhedron of Genus 5. *Advances in Geometry (to appear).*

[2] B. Grünbaum and G. C. Shephard. Polyhedra with Transitivity Properties. *C. R. Math. Rep. Acad. Sci. Canada*, 6(2):61–66, 1984.

[3] T. Tucker. Two Notes on Maps and Surface Symmetry. *Preprint.*