Unfolding Orthogrids with Constant Refinement*

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Abstract

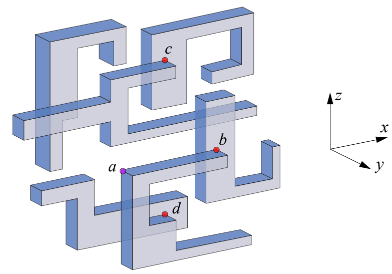
We define a new class of orthogonal polyhedra that can be unfolded without overlap with constant refinement of the gridded surface.

1 Introduction

An *unfolding* of a polyhedron is obtained by cutting the surface of the polyhedron and flattening it in the plane as a simple non-overlapping polygon. An *edge unfolding* considers only cuts made along edges, while general unfoldings allow cuts anywhere on the surface.

A polyhedron is *orthogonal* if each of its faces is perpendicular to a coordinate axis. An orthogonal polyhedron cannot be unfolded using edge cuts alone (the simplest example is a box sitting on top of a larger box). A *grid unfolding* of a polyhedron P allows additional cuts along grid edges created by slicing P with axis-aligned planes that pass through at least one polyhedron vertex, which we refer to as *grid planes*. Few nontrivial subclasses of orthogonal polyhedra are known to have grid unfoldings: orthotubes [BDD+98], orthostacks composed of orthogonally convex slabs [DM04], well-separated orthotrees [DFMO05] and terrains [O'R07].

A *k-refined grid unfolding*, defined for an orthogonal polyhedron and for some integer $k > 1$, allows additional cuts created by at most k planes parallel to and sandwiched between adjacent grid planes. A remarkable breakthrough is the k-refined grid unfolding of any orthogonal polyhedron homeomorphic to a sphere, where k is polynomial in the number of polyhedron vertices [DFO07]. For constant k, only two classes of orthogonal polyhedra are known to have a k-refined grid unfolding: orthostacks [BDD+98], and Manhattan towers [DFO08]. In this paper we introduce a new class of orthogonal polyhedra, which we call *orthogrids*, that have a k-refined grid unfolding for constant integer $k > 1$. An *orthogrid* is an orthogonal polyhedron of genus zero composed of extrusions of simple polygons (which we call *slabs*) stacked together so that a (left) vertex belongs to a unique slab. See Fig. below for an orthogrid example.



Our Result. We show that any orthogrid has a k-refined grid unfolding, for some constant $k > 1$. Our restriction to orthogrids is necessary to guarantee that some front/back surface piece is available at each vertex, so it can be used in unfolding as needed.

Definitions. Let $\mathcal{P}$ be an orthogrid. Classify the faces of $\mathcal{P}$ according to the direction of their outward normal: *front* $(+y)$, *back* $(-y)$, *top* $(+z)$, *bottom* $(-z)$, *right* $(+x)$ and *left* $(-x)$. We take the z-axis to define the vertical direction; *vertical* faces are parallel to the xz-plane or the yz plane.

Let $Y_1, Y_2, \ldots$ be planes passing through every vertex of $\mathcal{P}$, orthogonal to the y-axis. By convention, Y_1 is the plane with the largest y-coordinate. The portion of $\mathcal{P}$ bounded by planes Y_i and Y_{i+1} is composed of one or more disjoint slabs. The *front* (*back*) face of a slab is the face with normal pointing in the $+y$ $(-y)$ direction. The top, right, bottom and left faces surrounding a slab form a *band*. Clockwise (cw) and counterclockwise (ccw) directions are relative to a viewpoint at $y = \infty$.

2 Overview of Unfolding Algorithm

The main tool used in our unfolding algorithm is the single box unfolding introduced in [DFO08] and depicted in Fig. 1. The key idea is to identify a path (*spiral*) on the box surface that cycles around the four faces of the band at least once (see Fig. 1a); when flattened out, the spiral is an x-monotone orthogonal path (Fig. 1b) that covers the entire band

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when thickened (Fig. 1c). To help visualize this process, we marked the flattened spiral path in Fig. 1d with an L-shaped indicator (which we refer to as Hand) that always points in the direction of the x-axis in both $2D$ and $3D$.

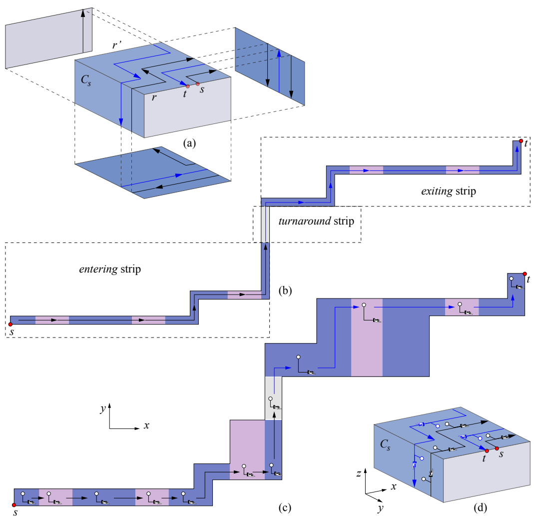

Figure 1: Box unfolding (a) Unfolding spiral. (b) Flattened spiral (c) Thickened spiral (d) 3D Hand

Algorithm Structure. The overall structure of the unfolding algorithm is as follows:

1. Compute a band unfolding tree T. Select the root of T to be a band of smallest y-coordinate, with ties broken arbitrarily.
2. Determine forward and return transition paths corresponding to each arc in T.
3. Starting with the root band, unfold each band as a conceptual unit, but interrupt the unfolding each time a forward transition path to a child is encountered. The unfolding follows the forward transition path to the child band, the child band is recursively unfolded, then the unfolding returns along the return transition path back to the parent, resuming the parent band unfolding from the point it left off. The resulting strip is laid out horizontally in the plane.
4. Attach the vertical front and back faces of $\mathcal{P}$ below and above appropriate horizontal sections of the unfolding strip from step 3.

The unfolding tree T has one node for each band and arcs connecting pairs of bands as follows: if two bands A and B intersect, select a leftmost topmost intersection point to be the "arc" connecting A and B. The graph thus obtained spans all bands, and T is an arbitrary spanning tree of this graph.

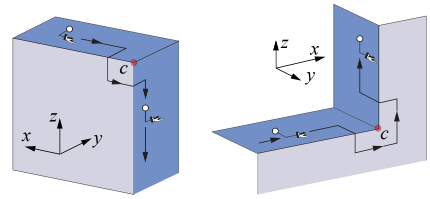

Figure 2: Forced horizontal-to-vertical turns.

The *forward and return transition paths* referenced in steps 2 and 3 of the algorithm use a small surface piece around the arc connecting a parent band to a child band. We seek to maintain two invariants throughout the unfolding procedure:

(I1) The Hand is parallel to the x-axis on a horizontal face of a band, pointing in the direction of the unfolding.

(I2) If at least one unfolded child is attached to a vertical edge e of a parent band A, then the Hand is parallel to the y-axis (pointing to either $+y$ or $-y$) while moving along e.

Invariant (I1) is necessary to maintain the x-monotonicity of the unfolding spiral, and invariant (I2) is necessary to facilitate the recursive unfolding of A's children attached to e. To maintain these two invariants, we sometimes need to enforce *turns* that change the Hand orientation at corner vertices. The horizontal-to-vertical turns are depicted in Fig. 2; the vertical-to-horizontal turns are symmetric. Note that the turns uses a small neighborhood around a corner vertex, hence the restriction orthogrids.

References

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