A Characterization of Consistent Digital Line Segments

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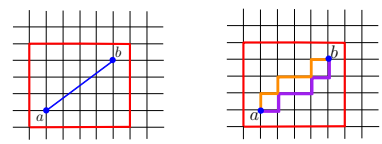
1 Abstract

Our main concern is the digitalization of line segments that satisfy a set of axioms naturally arising from Euclidean axioms which are called consistent digital line segments (CDS). Christ et al. [2] showed how to derive such a system of digital segments from a total order on the integers, and they left open the question on how to get a characterization of 2D CDSs. We resolve that question by giving a set of necessary and sufficient conditions for 2D CDSs. We also give a polynomial time algorithm which computes possibly many different total orders that satisfy the necessary and sufficient conditions that we give.

Motivation *Image segmentation* is the process of partitioning a digital image into multiple objects for better representation and analysis. We may wish to find an object in the digital image that has some geometric structure, and the definition of the object may depend on the definition of a digital line segment (e.g. a digital convex object, see Figure 1 (a)). Identifying a convex object depends on the definition of a line segment, but there may be many ways to define a digital line segment. In Figure 1 (b), we have two ways to define a digital line segment.

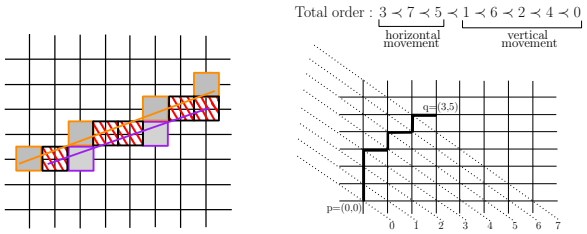


(a) (b)

Figure 1: Part (a) Line segment (a, b) in $\mathbb{R}^2$. Part (b) Line segment (a, b) in $\mathbb{Z}^2$.

Now, one trivial way to define the line segment is to take the pixels which are closest to the Euclidean line segment. See Figure 2 (a). For any two points $p, q \in \mathbb{Z}^2$, we let $S(p, q)$ denote the digital line segment from p to q. A key property that we want our segments to satisfy is the *subsegment property*, which is defined as - for each $r \in S(p, q)$, we have $S(p, r) \subset S(p, q)$. Trivial definitions (such as the one mentioned above) do not satisfy the subsegment property. See Figure 2 (a).

Chun et al.[1] was able to define consistent segments which all share an endpoint that do satsify the desired axioms. Later, Christ et al.[2] gave a definition for a CDS and left open the question of the full characterization of CDSs. We resolve the question by giving a set of necessary and sufficient conditions for CDSs.



(a) (b)

Figure 2: Part (a) An illustration of what we want to avoid. Part (b) A line segment from a point $p(0,0)$ to a point $q(3,5)$.

Preliminaries Christ et al.[2] derive the line segment from $p = (p^x, p^y)$ to $q = (q^x, q^y)$ from a total order on the integers as follows. Without loss of generality, assume that $p^x \leq q^x, p^y \leq q^y$. The segment starts from p, and it moves either up or right until it reaches q. If the line segment is at a point (x, y) for which $x + y$ is among the $q^y - p^y$ greatest elements in the interval $[p^x + p^y, q^x + q^y - 1]$ according to the total order, the line segment will go up. Otherwise, it will go right. See Figure 2 (b) for an example.

Necessary and Sufficient Conditions for CDSs In [2], every line segment is derived from the same total order. We now will assume that each point has its own total order, and we give a set of necessary and sufficient conditions that the total orders must satisfy. To help visualize these properties, we describe a "layout" where the intervals are placed on top of each other. We shift the two total orders left and right so that a single vertical line ℓ drawn through the total orders breaks both of them into the horizontal movement and vertical movement portions. See Figure 3. We call this ℓ the *dividing line*.

Now, Figure 4 (a) shows an illustration of the situation that we must avoid. a and b has opposite movements on the solid path and dotted path. These two paths violate the *subsegment property* of the Euclidean line segments. The dotted line goes up at a and goes right at b, so b is to the left of ℓ and a is to the right of ℓ. The opposite occurs for the solid line, and we have a is left of ℓ and b is right of ℓ. If this happens we call (a, b) a *bad pair*. See Figure 4 (b). If the subsegment property is violated, then there will be a bad pair. It follows that no bad pairs is a sufficient condition for CDSs, and we then show that this condition is also necessary. We then give a polynomial time algorithm which computes total orders without bad pairs.

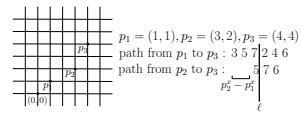
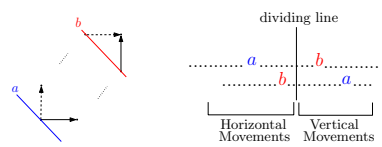


Figure 3: Layout view of total orders.



(a) (b)

Figure 4: Part (a) Dividing line and bad pair. Part (b) The situation if our properties are not met.

References

[1] T. J. Chun, M.Nollenburg. Consistent digital rays. *Computer Geometry*, 2009.

[2] M. S. Tobias Christ, Domotor Palvolgyi. Consistent digital line segments. In *Proceedings of the 2010 annual symposium on Computational geometry*, 2010.