New and Improved Stretch Factors of Yao Graphs

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Abstract

In this paper we study the stretch factors of Yao graphs. We prove that $Y_5$, the Yao graph with five cones, is a spanner with stretch factor $\rho = 2 + \sqrt{3} \approx 3.74$. Since $Y_5$ is the only Yao graph whose status of being a spanner or not was open, this completes the picture of the Yao graphs that are spanners: a Yao graph $Y_k$ is a spanner if and only if $k \geq 4$.

We also improve the known stretch factor of all the Yao graphs for odd $k > 5$ and reduce the known stretch factor of $Y_6$ from 17.7 to 5.8.

We complement the above results with a lower bound of 2.87 on the stretch factor of $Y_5$. We also show that $YY_5$, the Yao-Yao graph with five cones, is not a spanner.

1 Introduction

Let $S$ be a set of points in the plane. A geometric graph $G$ on the point set $S$ is called a $\rho$-spanner (or simply spanner if $\rho$ is a constant) if for every two points $a, b \in S$, the shortest path distance between $a$ and $b$ in $G$ is at most $\rho \cdot ||ab||$. The constant $\rho$ is called the stretch factor or spanning ratio of $G$.

For a fixed integer $k > 0$, the Yao graph [7] $Y_k$ is constructed by partitioning the space around each point $p \in S$ into $k$ equiangular cones, and connecting $p$ to a nearest neighbor in each cone. The Yao-Yao graph [6] $YY_k$ is constructed by augmenting the above construct with a second stage where each point keeps only the shortest incoming edge in each cone.

The spanning properties of Yao graphs have been extensively studied. It is known that $Y_2$ and $Y_3$ are not spanners [5], $Y_4$ is a spanner with stretch factor $8\sqrt{2}(29 + 23\sqrt{2})$ [2], $Y_6$ is a spanner with stretch factor 17.7 [4], and that for $k \geq 7$, $Y_k$ is a spanner with stretch factor $1 \left[1 - 2\sin\left(\frac{\pi}{k}\right)\right]$ [2]. The question of whether or not $Y_5$ is a spanner was previously open.

In this paper we prove that $Y_5$ is a $\rho$-spanner, where $\rho = 2 + \sqrt{3} \approx 3.74$. Combining this with the previous results, we now have a complete picture of the spanners that can be constructed with Yao graphs: any Yao graph $Y_k$ is a spanner if and only if $k \geq 4$.

We also improve the known stretch factor of all the Yao graphs for odd $k \geq 7$ to $\frac{1}{1 - 2\sin\left(\frac{3\pi}{4k}\right)}$ and reduce the known stretch factor of $Y_6$ to 5.8.

We complement the above result by giving a lower bound of 2.87 on the stretch factor of $Y_5$. We also show that $YY_5$, the Yao-Yao graph with five cones, is not a spanner.

Table 1 shows the current results on the stretch factors of Yao and Yao-Yao graphs for various values of the parameter $k$.

Table 1: Stretch factors of Yao and Yao-Yao graphs

<table>
<thead>
<tr>
<th>$k$</th>
<th>$Y_k$</th>
<th>$YY_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2, 3</td>
<td>$\infty$ [5]</td>
<td>$\infty$ [5]</td>
</tr>
<tr>
<td>4</td>
<td>$8\sqrt{2}(29 + 23\sqrt{2})$ [2]</td>
<td>$\infty$ [3]</td>
</tr>
<tr>
<td>5</td>
<td>3.74 [•]</td>
<td>$\infty$ [•]</td>
</tr>
<tr>
<td>6</td>
<td>5.8 [•]</td>
<td>$\infty$ [5]</td>
</tr>
<tr>
<td>$\geq 7$</td>
<td>$\frac{1}{1 - 2\sin\left(\frac{\pi}{k}\right)}$ even $k$ [2]</td>
<td>11.67 for $k = 6k'$, where $k' \geq 6$ [1]; Open for other values of $k \geq 7$</td>
</tr>
</tbody>
</table>
2 Our Results

We first prove an upper bound on the stretch factor of $Y_k$ for $k \geq 5$. For any two points $a, b \in S$, let $p(a, b)$ be the length of a shortest path in $Y_k$ from $a$ to $b$.

Lemma 1. Given three points $a$, $b$, and $c$, such that $|ac| \leq |ab|$ and $\angle bac \leq \alpha < 180^\circ$, then

$$bc \leq |ab| - (1 - 2 \sin(\alpha/2)) \cdot |ac|.$$  

Proof. Let $c'$ be the point on $ab$ such that $|ac| = |ac'|$. Since $acc'$ forms an isosceles triangle, $|cc'| = 2 \sin(\angle bac/2) \cdot |ac| \leq 2 \sin(\alpha/2) \cdot |ac|$. Now, by the triangle inequality, $|bc| \leq |cc'| + |c'b| \leq 2 \sin(\alpha/2) \cdot |ac| + |ab| - |ac'| = |ab| - (1 - 2 \sin(\alpha/2)) \cdot |ac|$. \qed

Theorem 1. For any odd integer $k \geq 5$, the $Y_k$-graph defined on a point set $S$ has stretch factor at most $t = 1/(1 - 2 \sin(30^\circ/8))$, where $\theta = 360^\circ/k$.

Proof Sketch. Let $a, b \in S$ be an arbitrary pair of points. Let $Q^b_a$ denote the cone with apex $a$ that contains $b$, and let $Q^a_b$ denote the cone with apex $b$ that contains $a$. Let $\alpha$ be the angle formed by the segment $ab$ with the bisector of $Q^b_a$, and let $\beta$ be the angle formed by $ab$ with the bisector of $Q^a_b$. Since $k$ is odd, the bisector of $Q^b_a$ is parallel to the right boundary of $Q^a_b$. Hence, we have that $\alpha = \theta/2 - \beta$. Assume without loss of generality that $\alpha$ is the smaller of these two angles. It follows that $\alpha \leq \theta/4$.

For the inductive step, if $a$ chooses $b$ as the nearest neighbor in $Q^b_a$, then $p(a, b) = |ab|$ and the proof is finished. So assume that $a$ chooses another point $c \neq b$ as the nearest neighbor in $Q^b_a$. Because $|ac| \leq |ab|$ and because $\angle cab \leq \theta/2 + \alpha \leq 30^\circ/4$, we can use Lemma 1 to derive $|cb| \leq |ab| - (1 - 2 \sin(30^\circ/8)) \cdot |ac| = |ab| - |ac|/t$, which is strictly smaller than $|ab|$. Apply the inductive hypothesis to $c, b$, we have a path between $a$ and $b$ of length: $p(a, b) \leq |ac| + t \cdot |cb| \leq |ac| + t \cdot (|ab| - |ac|/t) = t \cdot |ab|$. This completes the proof. \qed

Applying this result to $Y_5$ yields a stretch factor of $1/(1 - 2 \sin(22.5^\circ)) \approx 10.868$. Applying a more careful analyze, we further improve the stretch factor of $Y_5$:

Theorem 2. The $Y_5$ graph has stretch factor at most $2 + \sqrt{3} \approx 3.74$.

We also improve the stretch factor of $Y_6$.

Theorem 3. The $Y_6$-graph has spanning ratio at most $5.8$.

On the other hand, we give a point set whose $Y_5$ graph has stretch factor more than $2.87$. We also give a $YY_5$ graph whose stretch factor is unbounded.

References


Figure 1: Since opposite cones are not symmetric, either $\alpha$ or $\beta$ is small.

Proceed by induction on the distance $|ab|$. The base case holds because $ab \in Y_k$ when $|ab|$ is minimized.