New and Improved Stretch Factors of Yao Graphs

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Abstract

In this paper we study the stretch factors of Yao graphs. We prove that Y_5, the Yao graph with five cones, is a spanner with stretch factor $\rho = 2+\sqrt{3} \approx 3.74$. Since Y_5 is the only Yao graph whose status of being a spanner or not was open, this completes the picture of the Yao graphs that are spanners: a Yao graph Y_k is a spanner if and only if $k \geq 4$.

We also improve the known stretch factor of all the Yao graphs for odd $k > 5$ and reduce the known stretch factor of Y_6 from 17.7 to 5.8.

We complement the above results with a lower bound of 2.87 on the stretch factor of Y_5. We also show that YY_5, the Yao-Yao graph with five cones, is not a spanner.

1 Introduction

Let S be a set of points in the plane. A geometric graph G on the point set S is called a *ρ-spanner* (or simply *spanner* if ρ is a constant) if for every two points $a, b \in S$, the shortest path distance between a and b in G is at most $\rho \cdot ||ab||$. The constant ρ is called the *stretch factor* or *spanning ratio* of G.

For a fixed integer $k > 0$, the *Yao graph* [7] Y_k is constructed by partitioning the space around each point $p \in S$ into k equiangular cones, and connecting p to a nearest neighbor in each cone. The *Yao-Yao graph* [6] YY_k is constructed by augmenting the above construct with a second stage where each point keeps only the shortest incoming edge in each cone.

Table 1: Stretch factors of Yao and Yao-Yao graphs

k	Y_k	YY_k
$2,3$	∞ [5]	∞ [5]
4	$8\sqrt{2}(29+23\sqrt{2})$ [2]	∞ [3]
5	3.74 [·]	∞ [·]
6	5.8 [·]	∞ [5]
≥ 7	$\frac{1}{1-2\sin(\pi/k)}$ even k [2] $\frac{1}{1-2\sin(3\pi/4k)}$ odd k [·]	11.67 for $k = 6k'$, where $k' \geq 6$ [1]; Open for other values of $k \geq 7$

The spanning properties of Yao graphs have been extensively studied. It is known that Y_2 and Y_3 are not spanners [5], Y_4 is a spanner with stretch factor $8\sqrt{2}(29+23\sqrt{2})$ [2], Y_6 is a spanner with stretch factor 17.7 [4], and that for $k \geq 7$, Y_k is a spanner with stretch factor $\frac{1}{1-2\sin(\pi/k))}$ [2]. The question of whether or not Y_5 is a spanner was previously open.

In this paper we prove that Y_5 is a ρ-spanner, where $\rho = 2+\sqrt{3} \approx 3.74$. Combining this with the previous results, we now have a complete picture of the spanners that can be constructed with Yao graphs: any Yao graph Y_k is a spanner if and only if $k \geq 4$.

We also improve the known stretch factor of all the Yao graphs for odd $k \geq 7$ to $\frac{1}{1-2\sin(3\pi/4k)}$ and reduce the known stretch factor of Y_6 to 5.8.

We complement the above result by giving a lower bound of 2.87 on the stretch factor of Y_5. We also show that YY_5, the Yao-Yao graph with five cones, is not a spanner.

Table 1 shows the current results on the stretch factors of Yao and Yao-Yao graphs for various values of the parameter k.

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2 Our Results

We first prove an upper bound on the stretch factor of Y_k for $k \geq 5$. For any two points $a, b \in S$, let $p(a, b)$ be the length of a shortest path in Y_k from a to b.

Lemma 1. *Given three points a, b, and c, such that $|ac| \leq |ab|$ and $\angle bac \leq \alpha < 180°$, then*

$$|bc| \leq |ab| - (1 - 2\sin(\alpha/2)) \cdot |ac| \ .$$

Proof. Let c' be the point on ab such that $|ac| = |ac'|$. Since acc' forms an isosceles triangle, $|cc'| = 2\sin(\angle bac/2) \cdot |ac| \leq 2\sin(\alpha/2) \cdot |ac|$. Now, by the triangle inequality, $|bc| \leq |cc'| + |c'b| \leq 2\sin(\alpha/2) \cdot |ac| + |ab| - |ac'| = |ab| - (1 - 2\sin(\alpha/2)) \cdot |ac|$. □

Theorem 1. *For any odd integer $k \geq 5$, the Y_k-graph defined on a point set S has stretch factor at most $t = 1/(1 - 2\sin(3\theta/8))$, where $\theta = 360°/k$.*

Proof Sketch. Let $a, b \in S$ be an arbitrary pair of points. Let Q_a^b denote the cone with apex a that contains b, and let Q_b^a denote the cone with apex b that contains a. Let α be the angle formed by the segment ab with the bisector of Q_a^b, and let β be the angle formed by ab with the bisector of Q_b^a. Since k is odd, the bisector of Q_a^b is parallel to the right boundary of Q_b^a. Hence, we have that $\alpha = \theta/2 - \beta$. Assume without loss of generality that α is the smaller of these two angles. It follows that $\alpha \leq \theta/4$.

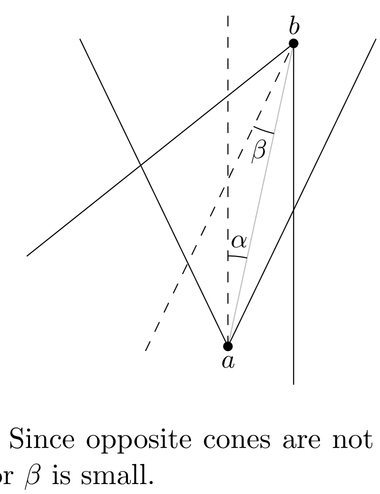

Figure 1: Since opposite cones are not symmetric, either α or β is small.

Proceed by induction on the distance $|ab|$. The base case holds because $ab \in Y_k$ when $|ab|$ is minimized.

For the inductive step, if a chooses b as the nearest neighbor in Q_a^b, then $p(a, b) = |ab|$ and the proof is finished. So assume that a chooses another point $c \neq b$ as the nearest neighbor in Q_a^b. Because $|ac| \leq |ab|$ and because $\angle cab \leq \theta/2 + \alpha \leq 3\theta/4$, we can use Lemma 1 to derive $|cb| \leq |ab| - (1 - 2\sin\frac{3\theta}{8}) \cdot |ac| = |ab| - |ac|/t$, which is strictly smaller than $|ab|$. Apply the inductive hypothesis to c, b, we have a path between a and b of length: $p(a, b) \leq |ac| + t \cdot |cb| \leq |ac| + t \cdot \left(|ab| - \frac{|ac|}{t}\right) = t \cdot |ab|$. This completes the proof. □

Applying this result to Y_5 yields a stretch factor of $1/(1 - 2\sin(27°)) \approx 10.868$. Applying a more careful analyze, we further improve the stretch factor of Y_5:

Theorem 2. *The Y_5 graph has stretch factor at most $2 + \sqrt{3} \approx 3.74$.*

We also improve the stretch factor of Y_6.

Theorem 3. *The Y_6-graph has spanning ratio at most 5.8.*

On the other hand, we give a point set whose Y_5 graph has stretch factor more than 2.87. We also give a YY_5 graph whose stretch factor is unbounded.

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