A Discrete Uniformization Theorem for Polyhedral Surfaces

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Abstract

The Poincaré-Koebe uniformization theorem for Riemann surfaces is a pillar in the last century mathematics. It states that given any Riemannian metric on a connected surface, there exists a complete constant curvature Riemannian metric conformal to the given one. The purpose of this paper is to introduce a discrete conformality for polyhedral metrics and discrete Riemann surfaces and establish a discrete uniformization theorem within the category of polyhedral metrics (PL metrics) on compact surfaces.

1 Introduction

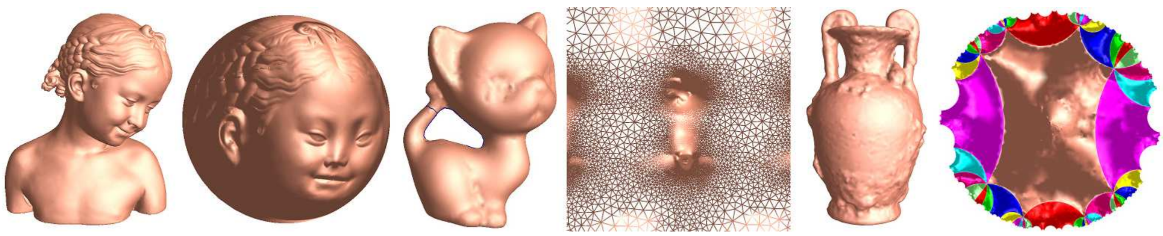
Given a closed surface S and a finite non-empty set $V \subset S$, we call (S, V) a *marked surface*. The objects of our investigation are *polyhedral metrics* (or simply PL metrics) on surfaces. By definition, a PL metric on (S, V) is a flat cone metric on S whose cone singularities are in V. The *discrete curvature* of a PL metric on (S, V) is the function on V sending a vertex $v \in V$ to 2π less the cone angle at v. A triangulation $\mathcal{T}$ of S with vertex set V is called a *triangulation* of (S, V).

Definition 1.1 (Discrete Conformality and Discrete Riemannian Surface) *Two PL metrics d and d' on (S, V) are discrete conformal if there exist sequences of PL metrics $d = d_1, \cdots, d_m = d'$ on (S, V) and triangulations $\mathcal{T}_1, \cdots, \mathcal{T}_m$ of (S, V) satisfying (1) each $\mathcal{T}_i$ is Delaunay at d_i; (2) if $\mathcal{T}_i = \mathcal{T}_{i+1}$, there exists a function $u : V \to \mathbb{R}$, called a* conformal factor, *so that if e is an edge in $\mathcal{T}_i$ with end points v and v', then the lengths $l_{d_{i+1}}(e)$ and $l_{d_i}(e)$ of e in d_i and d_{i+1} are related by $l_{d_{i+1}}(e) = l_{d_i}(e)e^{u(v)+u(v')}$, (3) if $\mathcal{T}_i \neq \mathcal{T}_{i+1}$, then (S, d_i) is isometric to (S, d_{i+1}) by an isometry homotopic to the identity in (S, V). The discrete conformal class of a PL metric is called a discrete Riemann surface.*

Our main theorem is as follows

Theorem 1.2 *Suppose (S, V) is a closed connected marked surface and d is any PL metric on (S, V). Then for any $K^* : V \to (-\infty, 2\pi)$ so that d' is discrete conformal to d and the discrete curvature of d' is K^*. Furthermore, the discrete Yamabe flow with surgery associated to curvature K^* with initial value d converges to d' exponentially fast.*

The similar theorem for hyperbolic cone metrics on (S, V) has been proved as well.



2 Teichmülller Space of PL metrics and decorated hyperbolic metrics

Two PL metrics d, d' on (S, V) are called *equivalent* if there is an isometry $h : (S, V, d) \to (S, V, d')$ so that h is iostopic to the identity map on (S, V). The *Teichmüller space of all PL metrics* on (S, V) denoted as $T_{pl}(S, V)$ is the set of all equivalence classes of PL metrics on (S, V), i.e.

$$T_{pl}(S, V) = \{d | d \; is \; a \; PL \; metric \; on \; (S, V)\}/isometry \cong id.$$

A *decorated hyperbolic metric* on (S, V) is a complete finite area hyperbolic metric d on (S, V) together with a horoball H_i centered at the i-th cusp at v_i for each i. We can parameterize it as (d, w), where w_i being the length of horocycle ∂H_i. Two decorated hyperbolic metrics are *equivalent* if there is an isometry h between them so that h is homotopic to the identity and h preserves horoballs. The space of all equivalence classes of decorated hyperbolic metrics on $S-V$ is defined to be the *decorated Teichmüller space* $T_D(S, V)$.

$$T_D(S, V) = \{d | d \; is \; a \; decorated \; hyperbolic \; metirc \; on \; (S, V)\}/isometry \cong id.$$

It is well known that $T_D(S, V) = T(S, V) \times \mathbb{R}^n_{>0}$, where $T(S, V)$ is the conventional Teichmüller space of complete hyperbolic metrics of finite area on $S - V$.

For a given triangulation $\mathcal{T}$ of Σ, $D_{pl}(\mathcal{T})$ is the set of all equivalence classes of PL metrics d in $T_{pl}(\Sigma)$ so that $\mathcal{T}$ is isotopic to a Delaunay triangulation of d, then $T_{pl}(S, V) = \cup_{[\mathcal{T}]} D_{pl}(\mathcal{T})$, where the union is over all isotopy classes $[\mathcal{T}]$ of triangulations of Σ. let $D(\mathcal{T})$ be the set of all equivalence classes of decorated hyperbolic metrics (d, w) in $T_D(\Sigma)$ so that $\mathcal{T}$ is isotopic to a Delaunay triangulation of (d, w). Penner proved that $T_D(\Sigma) = \cup_{[\mathcal{T}]} D(\mathcal{T})$.

3 Diffeomorphism between Teichmüller Spaces

Suppose $\mathcal{T}$ is a triangulation of (S, V) with $E = E(\mathcal{T})$. For each $x : E \to \mathbb{R}_{>0}$, such that for each face t with three edges e_i, e_j and e_k, the triangle inequality holds, then x gives a PL metric d on Σ. x is called the edge length coordinates, and denoted as $\phi_{\mathcal{T}} : \mathbb{R}^E \to T_{pl}(\Sigma)$. Similarly, for each $x : E \to \mathbb{R}_{>0}$, we can construct a decorated hyperbolic metric (d, w) on Σ. If t is a triangle in $\mathcal{T}$ with edges e_i, e_j, e_k, one replaces t by the decorated ideal triangle of edge lengths $2 \ln x(e_i), 2 \ln x(e_j)$ and $2 \ln x(e_k)$ and glues these decorated ideal triangles isometrically along the corresponding edges preserving decoration. One obtains a decorated hyperbolic metric (d, w). x is called the λ-length coordinates of the decorated hyperbolic metric (d, w), and denoted as and denoted as $\psi_{\mathcal{T}} : \mathbb{R}^E \to T_D(\Sigma)$.

Let $A_{\mathcal{T}} := \psi_{\mathcal{T}} \circ \phi_{\mathcal{T}} : D_{pl}(\mathcal{T}) \to D(\mathcal{T})$, the $A_{\mathcal{T}}$ is real analytic diffeomorphism. Let A be the gluing of $A_{\mathcal{T}}$, then

Theorem 3.1 *The homeomorphism $A : T_{pl}(S, V) \to T_D(S, V)$ is a C^1 diffeomorphism.*

Furthermore, discrete conformal PL metrics are mapped to $\{p\} \times \mathbb{R}^n_{>0}$. Let $u_i = \ln w_i$, define the discrete curvature map $F : \mathbb{R}^n \to (-\infty, 2\pi)^n$ by $F(u) = K_{A^{-1}(p,w(u))}$. The injectivity of F is proven using variational principle. The surjectivity of F is obtained from theorem 3.1 using domain invariance theorem. Details can be found in [1].

References

[1] Xianfeng Gu, Feng Luo, Jian Sun, and Tianqi Wu. A discrete uniformization theorem for polyhedral surfaces. *arXiv:1309.4175*, 2013.