Universality in Geometric Graph Theory

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Abstract. A *graph* represented by points (vertices) and line segments (edges) in Euclidean plane is called a *geometric graph*. Despite the severe constraint that all edges are straight-lines, geometric graphs are surprisingly flexible: every abstract graph has infinitely many geometric realizations, characterized by combinatorial and metric attributes such as edge crossings, edge lengths, angles, and areas. Their versatile applications in cartography, rigidity theory, and information visualization take advantage of this flexibility. This talk surveys recent results and open problems on universal structures for various types of geometric graphs.

A *universal point set* for n-vertex planar graphs is a set of points in Euclidean plane that can accommodate a crossing-free realization of every planar graph on n vertices. For $n \leq 10$, some sets of n points can accommodate all n-vertex plane graphs [Cardinal et al.] (see Fig. 1 for a small example), but every universal point set must have at least $(1.235 - o(1))n$ points [Kurowski]. The currently known smallest universal point sets have size $n^2/2$ [Bannister et al.]: this bound has been improved by only a factor of 2 over the last 23 years. Upper bounds of $O(n \log n)$ and $O(n^{3/2} \log n)$ have recently been proved in special cases and for certain relaxations of the problem, using partial orders and permutations.

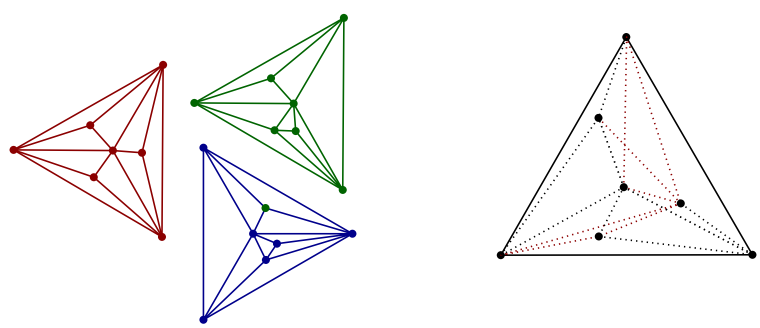

Figure 1: Three nonisomporphic planar three-trees on $n = 7$ vertices (left). All three graphs admit a geometric realization on a universal set of 7 points in the plane (right).

A planar graph G is *length universal* if every distribution of edge lengths on G can be realized in some geometric embedding of an arbitrary triangulation of G. Recently, we have classified all graphs that are length universal, using tools from rigidity theory and graph drawing [Fulek et al.].

The talk concludes with open problems about universal sets of lengths, slopes, face areas, and crossing angles of geometric graphs. Some of the simplest variants of these problems remain unsolved: Do any two planar graphs on n vertices have compatible realizations? Does a given graph have a geometric realization that satisfies given metric constraints (length, angle, or area)?

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