Simple Rectilinear Polygons are Perfect under Rectangular Vision

Esther Arkin[*†] Michael Biro[‡†] Omrit Filtser[§] Matthew J. Katz[§†] Joseph S.B. Mitchell [* †] Christiane Schmidt [* ¶]

Abstract

The Art Gallery problem (see O'Rourke [3] for an overview) asks for the minimum number, $g(P)$, of guards (points) that are necessary to see a given polygon P. Computing $g(P)$ is NP-hard. The maximum number, $w(P)$, of independent witness points within P is a lower bound on $g(P)$ ($w(P) \leq g(P)$); here, two points $w_1, w_2 \in P$ are *independent* if their visibility regions are pairwise-disjoint (i.e., no single guard can see both) [1]. In this paper we consider a special kind of visibility, *rectangular vision* (or *r-visibility* [4]), in which $p, q \in P$ *see* each other if the rectangle spanned by those two points, $r(p,q)$, is fully contained in P. Worman and Keil [4] show that $g(P)$ can be computed in polynomial time in rectilinear simple polygons P under rectangular vision.

We show that $g(P) = w(P)$ in rectilinear simple polygons under rectangular vision. This fact is a consequence of [4] (as shown in the full paper); it also follows by a direct geometric proof, given here, which may be simpler. Polygons for which $g(P) = w(P)$ have been called "*perfect*" (Mitchell), so our result can be interpreted as showing that rectilinear polygons under rectangular vision are perfect. In the full paper, we show that a corollary of this result is that $g(P)$ can be computed significantly more efficiently than previously known ($O(n^{17})$ [4]).

1 Notation and Preliminaries

We restrict ourselves to rectangular vision. We let $VP(p)$ denote the *visibility polygon* of point $p \in P$; note that $VP(p)$ is a union of rectangles anchored in p. Since we sometimes consider various polygons and subpolygons, when we want to emphasize that visibility is with respect to a particular polygon P, we will write $VP_P(p)$. For a given set of witness points W we refer to the *red/white decomposition* of P (RWD(P)): the visibility polygons of points in W are "red", while the remaining subpolygons of P are colored "white".

[*] estie@ams.stonybrook.edu, jsbm@ams.stonybrook.edu, cschmidt@ibr.cs.tu-bs.de; Department of Applied Mathematics and Statistics, Stony Brook University, USA.

[†] Partially supported by the National Science Foundation (CCF-1018388) and the US-Israel Binational Science Foundation (project 2010074).

[‡] Michael.J.Biro@williams.edu; Department of Mathematics and Statistics, Williams College, USA.

[§] matya@cs.bgu.ac.il, omritna@post.bgu.ac.il; Department of Computer Science, Ben-Gurion University of the Negev, Beer-Sheva, Israel.

[¶] Supported by the Postdoc Program of the DAAD (German Academic Exchange Service).

2 Perfectness

Theorem 1 *Rectilinear simple polygons are perfect ($w(P) = g(P)$) under rectangular vision.*

We give some lemmas before proving Theorem 1.

Lemma 2 *Let W and W' be two maximum-cardinality witness sets, each of size $k = w(P)$. Let G be the intersection graph of the visibility polygons of witnesses from W and W'. Then any connected component (CC) of G contains an equal number of elements from W and W'.*

Proof. Assume there is a CC of G with t elements from W and $t+1$ elements from W'. Take the latter $t+1$ witnesses and the $k-t$ witnesses from W that are in a different CC from G (and hence independent of all of these witnesses); together, they form a witness set of size $t+1+k-t = k+1$, a contradiction. □

Lemma 3 *For three points $w, p, q \in P$, if $VP(w) \cap VP(p) \neq \emptyset$ and $VP(w) \cap VP(q) \neq \emptyset$ and $VP(q) \cap VP(p) \neq \emptyset$, then $VP(w) \cap VP(p) \cap VP(q) \neq \emptyset$.*

Proof of Theorem 1. The proof is by induction on the number, k, of independent witnesses. The base case follows from Lemma 3, in combination with a theorem of Breen on families of orthogonally convex polygons [2].

Claim 4: (Base Case) $w(P) = 1 \Rightarrow g(P) = 1$.

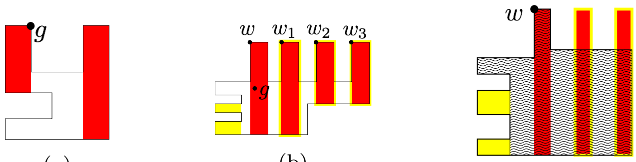
Induction Hypothesis: If P has k witnesses, P can be guarded by k guards. ($w(P) = k \Rightarrow g(P) = k$.)

Induction Step $k \to k+1$**:** Let P' be a polygon with $k+1$ witnesses.

The proof idea is to cover P' with two polygons: P_1, a polygon that is 1-guardable, and P_2', a possibly disconnected polygon that has at most k witnesses (w.r.t. visibility in P'), and, thus, by the induction hypothesis, is k-guardable. Hence, P' is $(k+1)$-guardable.

Claim 5: There exists an *ear witness*, i.e., a witness w such that all other witnesses $w_1, \ldots, w_k$ are in the same connected component of $P' \setminus VP(w)$, let this component be denoted by CC_k. (Let $W = \{w, w_1, \ldots, w_k\}$.)

Proof. Take RWD(P). Its dual is a tree, and it is naturally 2-colored by the decomposition. Define a root arbitrarily, and take the lowest red vertex (i.e., a red vertex whose only descendants, if any, are white vertices). Then, the witness corresponding to that vertex



(a) (b) (c)

Figure 1: (a) g located as shown results in $P' \setminus VP(g)$ allowing 2 witnesses. Thus, we need to include parts of CC_k to P_1. (b) Infeasible location for g. (c) P_2 indicated in yellow may be disconnected. We consider visibility in P' to exclude the possibility of more witnesses in the components.

is an ear witness, since removing it only separates unwitnessed white regions from the main polygon. $\square$

For the definition of P_1 we place a guard g in $VP(w)$, and define its visibility polygon to be P_1. To assure that the remaining polygon does not allow more than k witnesses, we cannot just place g on an arbitrary position in $VP(w)$; see Figure 1(a).

Let $U = \{u \notin VP(w) : VP(u) \cap VP(w) \neq \emptyset; VP(u) \cap VP(w_i) = \emptyset \forall i\}$. U collects all points that are privately seen from $VP(w)$ but from no other witness visibility polygon. Note that if such points exist, their visibility polygons must all intersect in a common point or region in $VP(w)$ (if $\exists u_1, u_2 \in U : VP(u_1) \cap VP(u_2) = \emptyset \Rightarrow \{u_1, u_2, w_1, \ldots, w_k\}$ witness set of size $k+2$ for P').

We now restrict the location of g to the set Q that monitors all private neighbors of w: $Q = \cap_{u \in U} VP(u)$ ($Q \neq \emptyset$). If $U = \emptyset$, then $Q = VP(w)$.

Again, the restriction of g to Q is not enough to achieve a feasible location (see Figure 1(b)): If we place g as indicated in the figure, and delete its visibility polygon, the remaining polygon indicated in yellow still allows 4 ($= k+1$) witnesses w.r.t visibility in P'.

Let $I_1 = \{p_1, \ldots, p_M\}$ be a maximum independent set (max IS) in $P' \setminus \{VP(w) \cup \mathrm{CC}_k\}$, $I_2 = \{q_1, \ldots, q_N\}$ be a max IS in CC_k, such that at least one of the $\mathrm{VP}(q_i)$ intersects $\mathrm{VP}(w)$. That is, I_2 is a max set of points in CC_k, whose visibility polygons (in P') are pairwise disjoint and at least one of them intersects $VP(w)$. Let $I_2' \subseteq I_2$, $I_2' = \{q_1, \ldots, q_R\}$, be the subset in the same connected component of the intersection graph of the VP's of W and I_2 as w. Let the stabbing number of the VP's of $I_1 \cup I_2'$ be s.

Claim 6: For any IS X of visibility polygons in $I_1 \cup I_2'$: $|\{v \in W : VP(v) \cap VP(x), x \in X\}| \geq |X|$. Therefore, if X is a maximum IS of visibility polygons in $I_1 \cup I_2'$ and $|X| = t$, then one can find a matching of size t in the red-blue intersection graph consisting of blue VP's from points in X and red VP's from points in W.

The proof uses Lemma 2 with sets W and X. With Lemma 3 it yields that all stabbing points for the visibility polygons from points in $I_1 \cup I_2'$ are located in red witness visibility polygons of witnesses in W.

Claim 7: $s = t$.

Proof. We can give an IS of size s: All s_1 points in $I_1 \cup I_2'$ that do not interfere with others are included. We need $s_2 = s - s_1$ points to stab the remaining, intersecting visibility polygons. For the associated bipartite intersection graph the stabbing number is equivalent to a minimum edge cover, EC, a set of edges that covers all vertices. There exists an IS of the size of the EC: each chosen edge has at least one adjacent vertex that is not covered (otherwise its deletion reduces the size of the EC). We include one of these vertices per EC edge. Altogether, we obtain a max IS of size $s_1 + s_2 = s$, thus, $t < s$ is impossible. As we have t independent visibility polygons, and need at least s points to stab them, we have $t \leq s$. $\square$

Now we place the guard g in Q: monitoring $\mathrm{VP}(w)$ and all the points assigned via the stabbing. $P_1 = VP(g)$.

Claim 8: P_1 is 1-guardable. (By construction.)

$$P_2 = \bigcup_{i=1}^{k} VP(w_i) \cup \left(\bigcup_{i=1}^{k} VP(VP(w_i)) \setminus P_1 \right)$$

P_2 may be disconnected. To not allow $k+1$ witnesses in P_2 we consider visibility in P'; see Figure 1(c).

Claim 9: $w(P_2) \leq k$ (with visibility defined in P').

Proof. Assume there is a witness set $W' = \{w_1', \ldots, w_{k+1}'\}$ of cardinality $k+1$ in P_2. Consider all $w_i' \in W'$ with $VP_{P'}(w_i') \cap VP_{P'}(w) \neq \emptyset$. W.l.o.g. let $w_1', \ldots, w_s'$ be all witnesses of W' in the same CC of the intersection graph of the visibility polygons of W and W'. Let the number of witnesses from W in this CC be r (w and $w_1, \ldots, w_{r-1}$).

$s = r - 1$. W' has $k+1$ witnesses, thus, $k+1-s$ are in a different CC. Thus, (by Lemma 2) $w, w_1, \ldots, w_{r-1}$, $w_{s+1}', \ldots, w_{k+1}'$ form an independent witness set of size $1+s+k+1-s = k+2$ for P', a contradiction.

$r < s$. Then, (by Lemma 2) $w_1', \ldots, w_s'$, $w_r, \ldots, w_k$ are an independent witness set of size $> s+(k+1-s) = k+1$ for P', a contradiction.

$r = s$. We use as many witnesses in the CC of w as are maximally stabbed. Thus, one of $w_1', \ldots, w_s'$ is located in P_1. A contradiction. $\square$

Finally, we note (without proof due limited space):

Claim 10(Complete coverage): $P' = P_1 \cup P_2$.

References

[1] Y. Amit, J. S. B. Mitchell, and E. Packer. Locating guards for visibility coverage of polygons. *Int. J. Comput. Geometry Appl.*, 20(5):601–630, 2010.

[2] M. Breen. A helly-type theorem for simple polygons. *Geometriae Dedicata*, 60(3):283–288, 1996.

[3] J. O'Rourke. *Art Gallery Theorems and Algorithms.* International Series of Monographs on Computer Science. Oxford University Press, New York, NY, 1987.

[4] C. Worman and J. M. Keil. Polygon decomposition and the orthogonal art gallery problem. *Int. J. Comput. Geometry Appl.*, 17(2):105–138, 2007.