Golden Triangulations

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Many geometric gems are found in the most elementary problems. We would like to begin our discussion of golden triangles with such a problem.

Elementary Problem: *Characterize all isosceles triangles which can be "split" into two isosceles triangles by a single line through one of it's vertices, so that at least one of the sub-triangles is similar to the original triangle.*

It is always possible to split a isosceles triangle at it's largest angle so that one of the triangles in the resulting two sub-triangles is similar to the original isosceles triangle. However, in general, the second sub-triangle is not isosceles. It is easy to check that the second sub-triangle is isosceles if and only if the angle measurements of the original triangle are $(36, 72, 72)$, $(90, 45, 45)$, or $(108, 36, 36)$.

Starting with a triangle with angle measurements $(90, 45, 45)$, if we split the 90 degree angle, as above, then both sub-triangles clearly have measurements $(90, 45, 45)$. We observe that the ratio of the longer side to the shorter side, in such a triangle, is $\sqrt{2}$.

What is quite interesting is that the other two triangles are a pair, in that when we do this "split" we produce one triangle similar to the original and the other isosceles triangle is its pair. Therefore this splitting technique can be carried out an infinite number of times on the resulting isosceles triangles formed by the previous split so that every triangle created is similar to one of the triangles of the pair. It is also interesting that the ratio of the longer side to the shorter side in both of these triangles is $\frac{1+\sqrt{5}}{2}$, or the golden ratio.

Definition 1 *A **golden triangle** is an isosceles triangle such that the ratio of the longer side length to the shorter side length is τ, the golden ratio.*

Remark 1 *Every golden triangle will have as its angles measures either $(36, 72, 72)$ or $(108, 36, 36)$. Such triangles are also known as Penrose-Robinson triangles [8].*

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With the above connection it is natural to ask for which point set does there exist a partitioning, or more specifically a triangulation, where every region is a golden triangle. In such a region it would then be natural to refine the partition by this splitting process.

Definition 2 *A triangulation T is a **golden triangulation** if for every $t \in T$, t is a golden triangle.*

There are some basic observation which can be made about the measurements of the golden triangles of a golden triangulation. If one triangle of such a triangulated region has one of its side lengths (resp. its area) equaling an integer power of τ, then every triangle in the triangulation has side lengths (resp. area) equaling an integer power of τ. More is true. If one triangle of such a region has one of its side lengths (resp. its area) equaling an integer power of τ, then each boundary segment (resp. the area of the region) is a $\tau-$rational number, i.e. a sum of distinct integer powers of τ.

Turning our attention to polygonal regions, we will classify all convex polygons which can be triangulated with golden triangles and determine which can be done in a minimal sense.To do so we will need to categorize polygons by their interior angles.

Definition 3 *For a simple polygon P on n vertices, begin at a vertex of P and list each angle measure in a clockwise direction so that the n angles are listed as an n-tuple. The n-tuple will be called an **angle sequence** of the polygon P and is a representation of P.*

Definition 4 *Given a simple polygonal region P on n vertices which only has angle measures 36, 72, 108, or 144 let its **angle multiplier sequence** be a sequence of n terms from the set $\{1, 2, 3, 4\}$ obtained from dividing each term in the angle sequence by 36.*

Definition 5 *If there exists a simple polygonal region P having a golden triangulation, then its angle sequence is called **admissible**. (Similarly the corresponding angle multiplier sequence is called admissible.)*

Theorem 1 *Every sequence of length n, for $3 \leq n \leq 10$, such that the sum of the digits is $5(n-2)$ is admissible*

We will also classify all convex polygons which can be minimally golden triangulated, or triangulated by $n-2$ golden triangles.

For golden triangulations, we consider refinements that can be obtained by successively splitting golden triangles into two smaller golden triangles, as described above. One must be careful with this procedure, since splitting can result in a partition that is not a triangulation. In [5] the partitions that result from splitting golden triangles are shown to have application to the study of quasicrystals. We put the further restriction that the splitting process must result in a proper refinement with golden triangles. Such a refinement is called a **golden split refinement**.

Using the process outlined in Figures 1 and 2, applied to golden triangles, we can trivially find a golden split refinement of any golden triangulation.

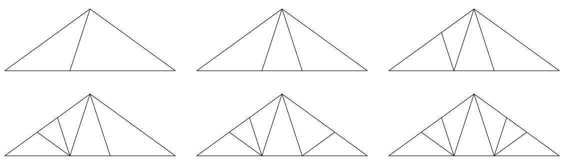

Figure 1:

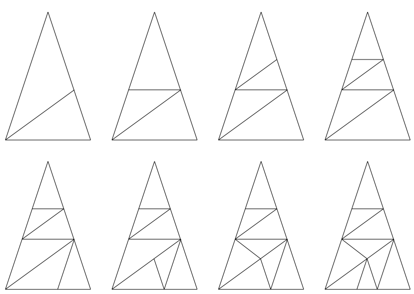

Figure 2:

Assuming that there exists a golden triangulation of n triangles, this refinement process obviously yields no fewer than $7n$ triangles and can produce as many as $9n$ triangles. In many cases we can do better. Nevertheless, it can be shown that any golden split refinement algorithm will refine the golden triangulation in Figure 3 will produce at least 16 triangles.

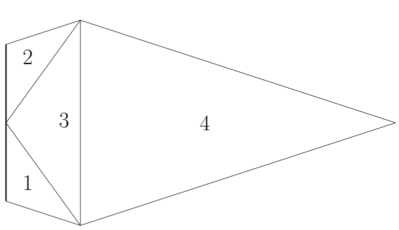

Figure 3:

We will also analyze the set of vertices that are generated by using a golden split refinement algorithm repeatedly on a given golden triangulation. We show that if we apply a golden split refinement algorithm to a single golden triangle, then each new vertex generated must have, with respect to the vertices of the original triangle, barycentric coordinates that are $\tau-$rational numbers. This naturally leads one to look for methods of generating golden triangulations from a given set of vertices which have $\tau-$rational barycentric coordinates with respect to some fixed golden triangle.

It is also easy to relate golden triangulations to Delauney Triangulations. If a golden triangulation does not have any occurrences of two adjacent obtuse golden triangles sharing their long edges, then the triangulation coincides (up to flips) with a Delaunay triangulation of its set of vertices.

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