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Zak Transform and a New Approach to Waveform Design

The ambiguity function $A_f(\tau,\nu)$ of a transmitted signal $f(t)$ measures the uncertainty with which the returning echo distinguishes, simultaneously, both ranges and velocities of a target system. Generally speaking, $A_f(\tau,\nu)$ is desired to be of "thumbtack" shape, i.e., a function whose absolute value has a graph with a strong peak at the origin over a broad shallow base. The ambiguity function can be computed directly from the Zak transform $Z_f(x,y)$ of the signal $f(t)$, so waveforms with desirable ambiguity functions can be designed in the Zak domain. In the Zak domain, computation of $A_f(\tau,\nu)$ on the integer lattice is exceptionally simple, particularly for pulse train signals. For a pulse train, the Zak transform is gotten by multiplying the Zak transform of a rectangular pulse of duration 1 by a multivariate trigonometric polynomial whose coefficients are the coefficients defining the pulse train. Reversing this observation, one can start with such a trigonometric polynomial and construct a pulse train signal. We propose a systematic method for constructing such waveforms, which we illustrate in a particular case.

I. INTRODUCTION

Designing signals with thumbtack ambiguity functions, i.e., functions whose absolute value has a graph with a strong peak at the origin over a broad shallow base, is a special case of the more general issue of designing signals with a prescribed ambiguity function. The many attacks on this difficult problem [14, p. 125] since the publication of Woodward's book have yielded a great deal of insight into the nature of the ambiguity function (see [4]), but no computationally practicable solution to the general synthesis problem has been provided. The elegant paper of Wilcox [13] provides a mathematically complete solution, but it should be borne in mind that the speed of convergence of his solution depends on the smoothness of the underlying functions. Since the ideal ambiguity function is a delta function, and hence a poor input to the Wilcox algorithm, the search for practical solutions to the synthesis problem remains a challenging problem. Algebraic properties of the mapping that maps a function $s(t)$ into its ambiguity function were studied by Auslander and Tolimieri [3].

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Additional properties of the ambiguity function were described by Rihaczec [12] and Grünbaum [8].

Among the most successful of the efforts along the above lines are certain frequency-hopping waveforms found by Costas (see [5]) which have thumbtack-like ambiguity surfaces. The Costas waveform construction was algebraic, relying on the construction of matrices having special combinatorial properties.

If one's goal is the development of a large class of waveforms, such algebraic methods prove to be restrictive. For example, while the number of $N \times N$ Costas arrays is quite large (2160 when $N = 10$), any two of the same size must necessarily share an ambiguity sidelobe location when $N > 3$ [6]. Also the number of cross-ambiguity coincidences between arrays both of size $N \times N$ can be quite large and depends strongly on the number-theoretic properties of N [10]. Our goal is to look at this problem using generalized pulse train signals that are generated by analytic, rather than algebraic, methods.

We remark that Rendas and Moura [11] have recently introduced a modified definition of the ambiguity function tailored to passive radar problems, wideband signals, and model mismatches. Under the narrowband assumption operative in our work, their definition becomes the classical Woodward definition. As they remark in their paper, the same tools cannot be used to study passive and active systems, and the Woodward definition, which we use, appears to be the appropriate assumption in our work.

II. ZAK TRANSFORM AND AMBIGUITY FUNCTION

As a tool for constructing such waveforms we use the Zak transform [9, 2]

$$Z_f(x,y) = \sum_{k=-\infty}^{\infty} f(x+k)e^{-2\pi iky}, \qquad k \in \mathbf{Z} \tag{1}$$

which changes the formation of pulse trains

$$f(t) = \sum_{m,n} a_{mn}\chi_{[0,1]}(t-m)e^{2\pi int} \tag{2}$$

in the "signal space" to multiplication

$$Z_f(x,y) = P(x,y)\cdot Z_{\chi_{[0,1]}}(x,y) \tag{3}$$

by a doubly-periodic function

$$P(x,y) = \sum_m \sum_n a_{mn} e^{2\pi i(nx+my)} \tag{4}$$

in "Zak space" ($\chi_{[0,1]}$ is the indicator function of the unit interval).

The significance of this for the problem of creating a particular ambiguity surface is that we can compute the ambiguity function

$$A_f = \int_{-\infty}^{\infty} f(t)\overline{f(t-\tau)}e^{2\pi i\nu t}\,dt \tag{5}$$

of a signal f directly from Z_f (cf. [2]), using the formula

$$A_f(\tau,\nu) = \int_0^1 \int_0^1 Z_f(x,y)\overline{Z_f(x+\tau,y+\nu)}e^{-2\pi ix\nu}\,dx\,dy. \tag{6}$$

Note that in terms of the Zak transform, the ambiguity function on the integer lattice is

$$A_f(n,m) = \int_0^1 \int_0^1 |Z_f(x,y)|^2 e^{2\pi i(mx+ny)}\,dx\,dy \tag{7}$$

so the $A_f(n,m)$s are the Fourier coefficients of the real, nonnegative function $|Z_f(x,y)|^2$.

Obviously, if $|Z_f(x,y)|^2 = 1$ on $[0,1]\times[0,1]$ then the signal $f(t)$ has an ambiguity function which is thumbtack on the integer lattice. The trivial example that produces such a $Z_f(x,y)$ is a rectangular pulse of duration 1. Recall ([4]) that the volume property of the ambiguity function requires that the volume under the graph of $|A_f(\tau,\nu)|^2$ must equal the square of the maximum, which means that the ambiguities cannot be pushed down too much, and in the case of the rectangular pulse, this constraint gives rise to undesirably high sidelobes.

From the above considerations, it is evident that waveform design can be carried out in the Zak domain, and in what follows, we describe one approach, suggested by this general philosophy.

III. WAVEFORM DESIGN

Our example makes use of a particular relation between Chebyshev polynomials [1]. In more detail, recall that if T_k is the kth Chebyshev polynomial ($T_k(x) = \cos(k \arccos x)$) then $T_k^2 = \frac{1}{2}(T_{2k}+1)$.

This suggests setting $Z_f(x,y) = T_k(\varphi(x,y))$ on $[0,1]\times[0,1]$, for some real-valued function $\varphi(x,y)$, since then $|Z_f|^2 = T_k^2 = \frac{1}{2}(T_{2k}+1)$ (here we have used the fact that $|Z_{\chi_{[0,1]}}|^2 = 1$). Thus, by (7),

$$A_f(n,m) = \frac{1}{2}\int_0^1 \int_0^1 T_{2k}(\varphi(x,y))e^{2\pi i(mx+ny)}\,dx\,dy + \tfrac{1}{2}\delta(0,0). \tag{8}$$

Thus the question now is to find $\varphi(x,y)$ leading to good thumbtack properties of the ambiguity function $A_f(\tau,\nu)$ for general parameters τ, ν as well as for integer parameters $\tau = n$, $\nu = m$. In what follows, we discuss in detail a particular class of examples, and the corresponding ambiguity functions. We remark that the discussion corresponding to values on the integer lattice is in some sense simpler.

An intriguing possibility is to take $\varphi(x,y)$ to be a radial function, i.e., a function of $r = \sqrt{x^2+y^2}$, which leads to a computationally one-dimensional problem. An empirically good choice of this type is provided, e.g., by $\varphi(x,y) = \cos(2\pi r)$, which corresponds to

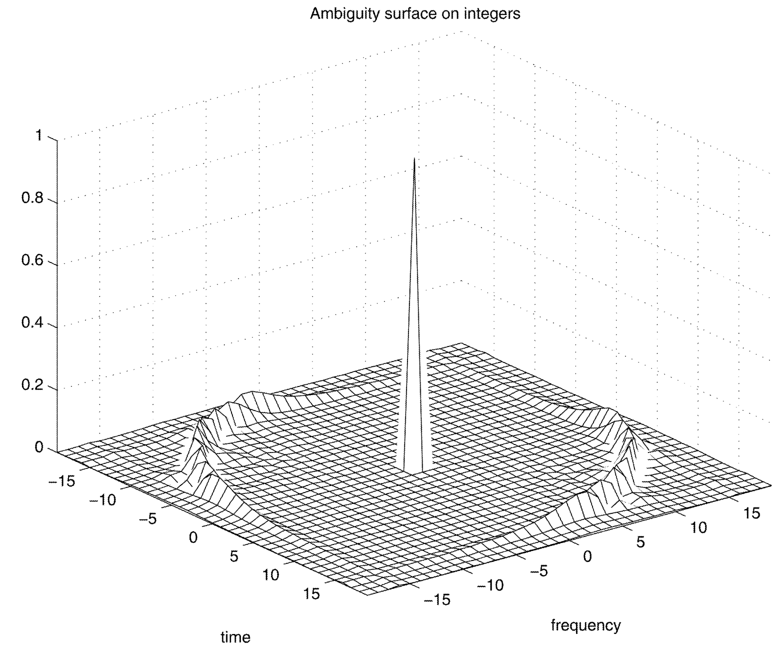

Fig. 1.

$Z_f(x,y) = T_k(\cos(2\pi\sqrt{x^2+y^2})) = \cos(2\pi k\sqrt{x^2+y^2})$, and $T_{2k} = \cos(4\pi k\sqrt{x^2+y^2})$. The trigonometric polynomial (4) needed to form the pulse train (2) is obtained by standard numerical approximations

$$\sum_n \sum_m a_{mn} e^{2\pi i(nx+my)} \approx \cos(2\pi k\sqrt{x^2+y^2}). \qquad (9)$$

In this example, the preliminary composition with a cosine reduces the functional equation for the Chebyshev family to the double angle formula for the cosine, and in general, Chebyshev functions can be replaced by any family of functions possessing a functional equation of the necessary type. However, preliminary analysis suggests advantages to our choice of the Chebyshev system. In the example, we have deliberately precomposed with a cosine, among other things because it brings out very clearly the near-term cancellation and hence the tendency to zero of the integral in (8), but the scope of our method, which is determined by the wealth of available choices of φ, extends well beyond the class of examples of this type. Additionally, we expect that in many cases we will be able to analytically derive precise descriptions of favorable zones for the corresponding ambiguity function.

Fig. 1 illustrates the ambiguity surface on the integer lattice for this choice of $Z_f(x,y)$ with $k = 8$. The guiding philosophy in this construction is to push some of the volume under the graph of the ambiguity function out to the complement of a good-sized disk about the origin (see (8)), thus leaving as little of the volume inside the disk as possible, subject to the limits imposed by the volume property of the ambiguity function.

The radius of the disk that can be achieved via this construction depends on k. Fig. 2 represent the ambiguity surface for continuous variables. In computing this surface, we have approximated the exact cosine on the right-hand side of (9) by a partial sum of its Fourier series, and then inserted the corresponding Fourier coefficients into the pulse train formula (2), in order to compute (5).

One of the interesting features of this example is the simplicity of the design. The trigonometric form of the Zak transform (on the unit square)

$$Z_f(x,y) = \sum_m \sum_n a_{mn} e^{2\pi i(nx+my)} = P(x,y) \qquad (10)$$

has the same coefficients a_{mn}s as a generalized pulse train

$$f(t) = \sum_{m,n} a_{mn}\chi_{[0,1]}(t-m)e^{2\pi int}. \qquad (11)$$

Numerical calculations show that in this case, the matrix $A = \{a_{mn}\}$ of coefficients in the pulse train has one dominant non-zero element a_n in each row and therefore the small components of the matrix A can be neglected, which effectively makes it into a matrix with only one non-zero element in each row. This makes our waveform f a frequency hop pulse train signal

$$f(t) = \sum_{n=1}^{2k+1} a_n \chi_{[0,1]}(t-n)e^{2\pi i\theta_n t} \qquad (12)$$

where $\theta = \{\theta_1, \theta_2, \ldots, \theta_{2k+1}\}$ is a set of the integers that range from $-k$ through k and k is an arbitrary integer (same as in (9)). Thus, given an integer k we can construct the frequency hop pulse train (12) of $2k+1$ subpulses with frequencies from $-k$ to k.

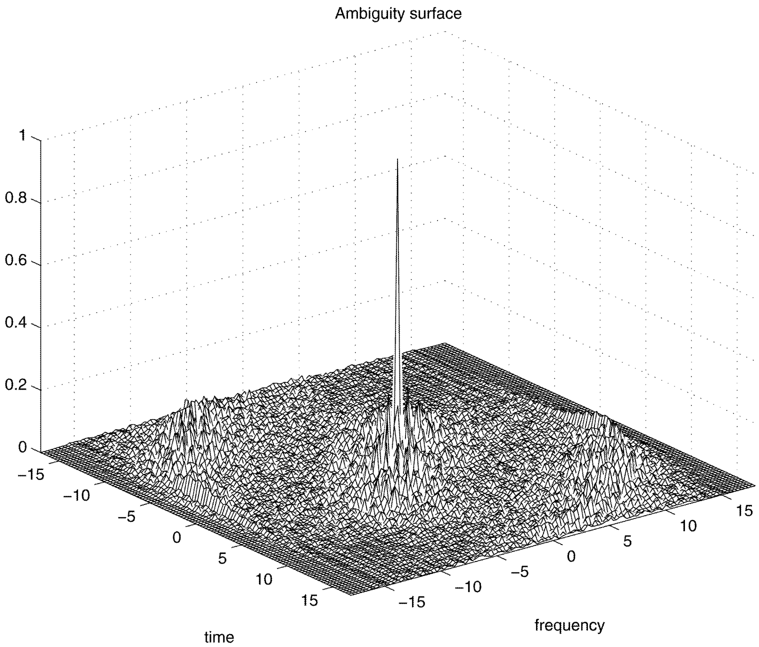

Fig. 2.

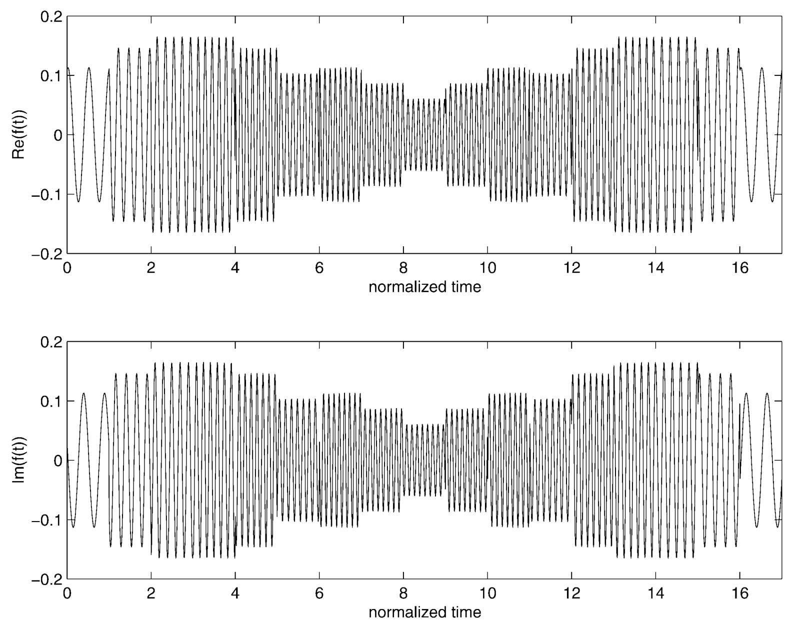

Fig. 3.

The real and imaginary parts of the frequency hopping waveform $f(t)$ are shown in Fig. 3. The matrix $A = \{a_{mn}\}$ of coefficients in the pulse train is given in Fig. 4, where black corresponds to a non-zero a_{mn}.

Fig. 5 represents the ambiguity surface of this frequency hopping waveform.

Another interesting feature of such an example is that a detailed analysis of favorable zones (low sidelobes) of the ambiguity surface for continuous values of τ and ν is possible. If we write the ambiguity function $A_f(\tau,\nu)$ as the sum of $A^{(1)}(\tau,\nu)$ and $A^{(2)}(\tau,\nu)$, where

$$A^{(1)}(\tau,\nu) = \sum_{m,n} |a_{mn}|^2 e^{-2\pi i(\nu m-\tau n)} A_{\chi[0,1]}(\tau,\nu) \quad (13)$$

$$A^{(2)}(\tau,\nu) = \sum_{\substack{m,n,k,l\\ k\neq m, l\neq n}} a_{mn}\overline{a_{kl}} e^{-2\pi i m(\nu-n)} e^{2\pi i l(\tau-m)} \times A_{\chi[0,1]}(\tau+(k-m),\nu+(l-n)) \quad (14)$$

we can easily see that the first term will produce the mainlobe of the ambiguity function and the second term will produce the sidelobes. In the ideal

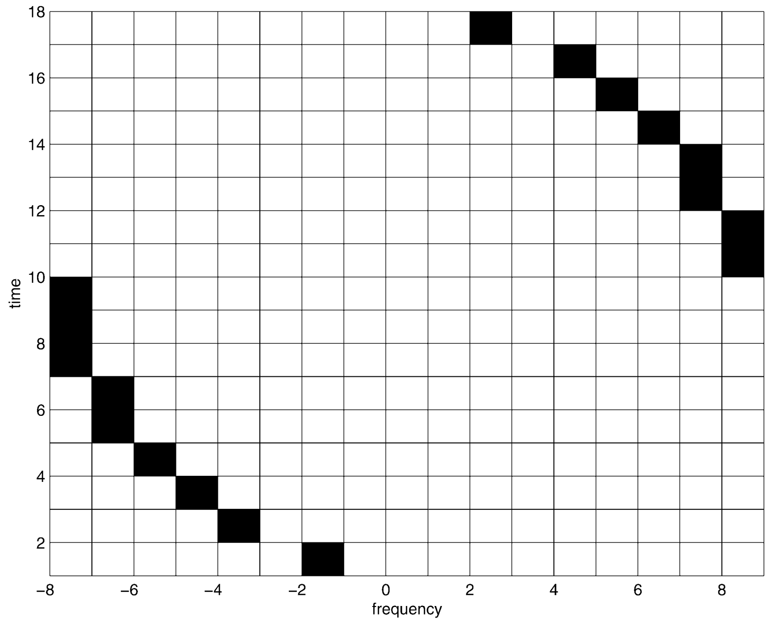

Fig. 4.

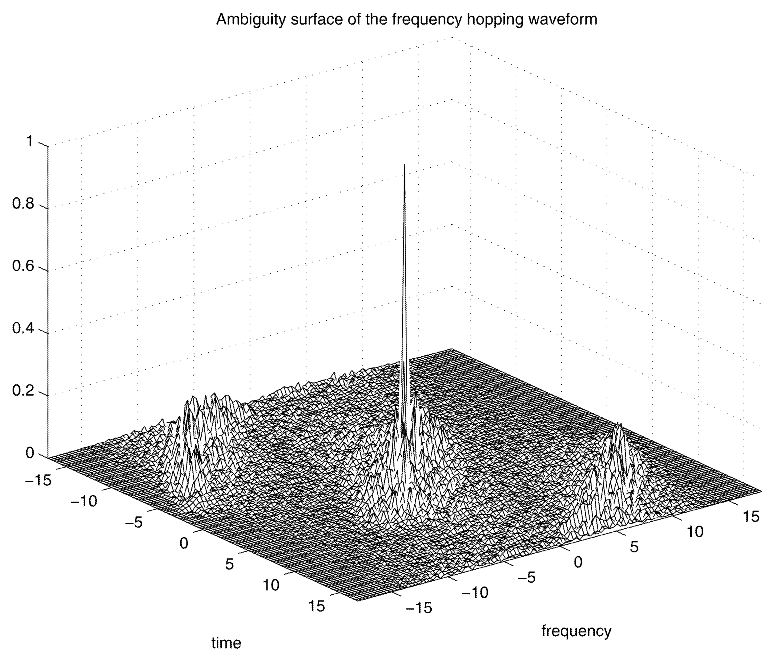

Fig. 5.

(continuous) case, the counterparts of the coefficients a_{mn} turn out to be located on a circle, and hence there are at most two hits between the matrix A and its shifted version. The areas where these hits may occur are illustrated in Fig. 6. The darker portions of the ambiguity image, except for the origin $(0,0)$, correspond to undesirable sidelobes. The analytical description of the boundaries of the sidelobe (darker) regions is given by $\tau^2 + \nu^2 < 4k^2$, $(\tau - k)^2 + \nu^2 > k^2$, $\tau^2 + (\nu - k)^2 > k^2$, $\tau > 0$, $\nu > 0$ for the positive time-frequency quadrant, $\tau^2 + \nu^2 < 4k^2$, $(\tau + k)^2 + \nu^2 > k^2$, $\tau^2 + (\nu + k)^2 > k^2$, $\tau < 0$, $\nu < 0$ for the negative time-frequency quadrant, $\tau^2 + (\nu - k)^2 < k^2$, $(\tau + k)^2 + \nu^2 < k^2$, or $\tau^2 + (\nu + k)^2 < k^2$, $(\tau - k)^2 + \nu^2 < k^2$ for the two parts of the central region.

It is interesting to note that the arcs which comprise the boundary of the shape in Fig. 6 correspond to circles of radius k and $2k$, respectively, and hence the geometry of Fig. 6 is k-invariant.

We are currently in the process of investigating various further possibilities for the selection of suitable $\phi(x,y)$s.

A more detailed description of some of the algorithms, as well as further examples of our method can be found in [7], which is available from the author.

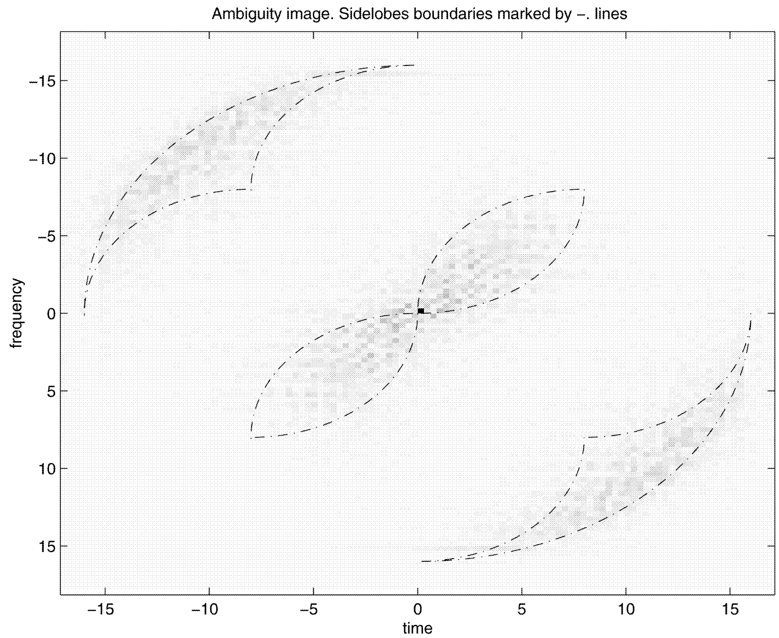

Fig. 6.

IV. CONCLUSION

Our decision to design waveforms in the Zak domain has led to the creation of design whose analysis over both the continuous and integer lattice variables suggests that they hold considerable promise.

As we have demonstrated, in the Zak domain, computation of $A_f(\tau,\nu)$ on the integer lattice is exceptionally simple, particularly for pulse train signals. We have seen that for a pulse train, the Zak transform is gotten by multiplying the Zak transform of a rectangular pulse of duration 1 by a multivariate trigonometric polynomial whose coefficients are the coefficients defining the pulse train. The reversal of this observation leads to the construction of a pulse train signal which comes from a trigonometric polynomial. The search for appropriate trigonometric polynomial for this purpose remains an ongoing goal.

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Errata: "Relative Targeting Architectures for Captive-Carry HIL Missile Simulator Experiments"[1]

Equation (4) should read

$$T(R_a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(R_a) & \sin(R_a) \\ 0 & -\sin(R_a) & \cos(R_a) \end{bmatrix} \tag{4}$$

[1] P. E. Pace, M. D. Nash, D. P. Zulaica, A. A. Di Mattesa, and A. Hosmer, *IEEE Transactions on Aerospace and Electronic Systems*, **37**, 3 (July 2001), 810–823.

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