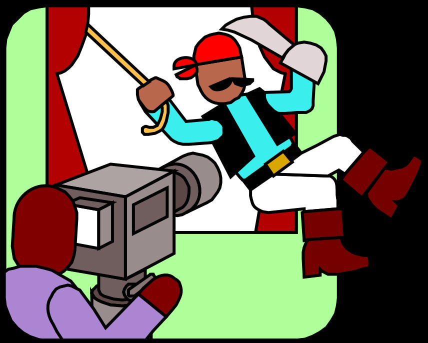
CSC I6716
Fall 2010



Topic 1 of Part II
Camera Models

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3D Vision

- Closely Related Disciplines
  - Image Processing – images to mages
  - Computer Graphics – models to images
  - Computer Vision – images to models
  - Photogrammetry – obtaining accurate measurements from images
- What is 3-D (three dimensional) Vision?
  - Motivation: making computers see (the 3D world as humans do)
  - Computer Vision: 2D images to 3D structure
  - Applications : robotics / VR /Image-based rendering/ 3D video
- Lectures on 3-D Vision Fundamentals
  - Camera Geometric Models (3 lectures)
  - Camera Calibration (3 lectures)
  - Stereo (4 lectures)
  - Motion (4 lectures)

Lecture Outline

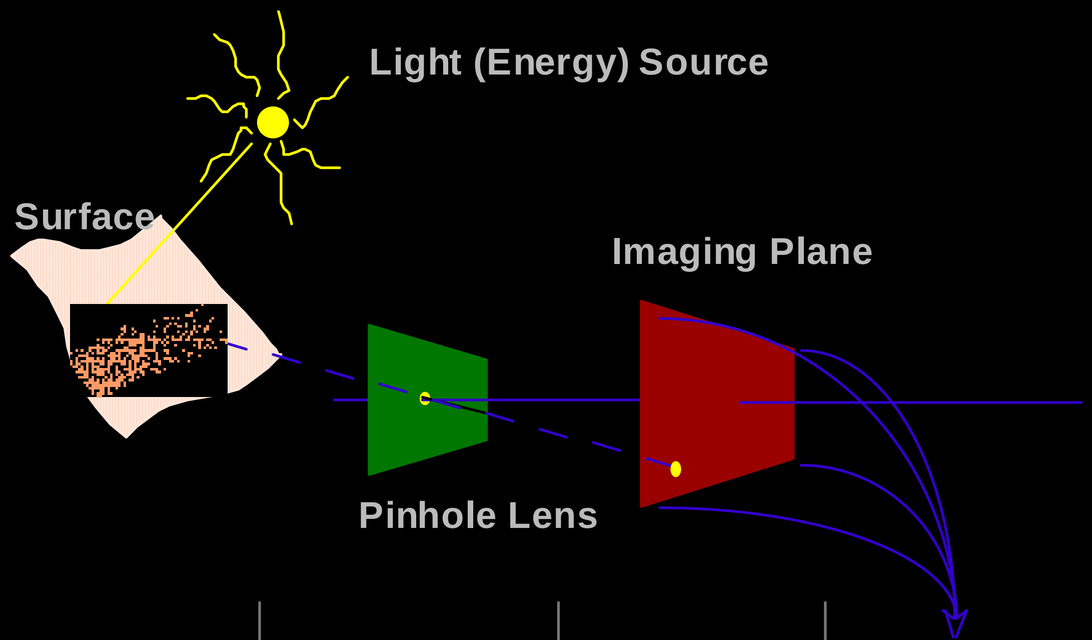
- Geometric Projection of a Camera
  - Pinhole camera model
  - Perspective projection
  - Weak-Perspective Projection
- Camera Parameters
  - Intrinsic Parameters: define mapping from 3D to 2D
  - Extrinsic parameters: define viewpoint and viewing direction
    - Basic Vector and Matrix Operations, Rotation
- Camera Models Revisited
  - Linear Version of the Projection Transformation Equation
    - Perspective Camera Model
    - Weak-Perspective Camera Model
    - Affine Camera Model
    - Camera Model for Planes
- Summary

Lecture Assumptions

- Camera **Geometric** Models
  - Knowledge about 2D and 3D geometric transformations
  - Linear algebra (vector, matrix)
  - This lecture is only about geometry
- Goal

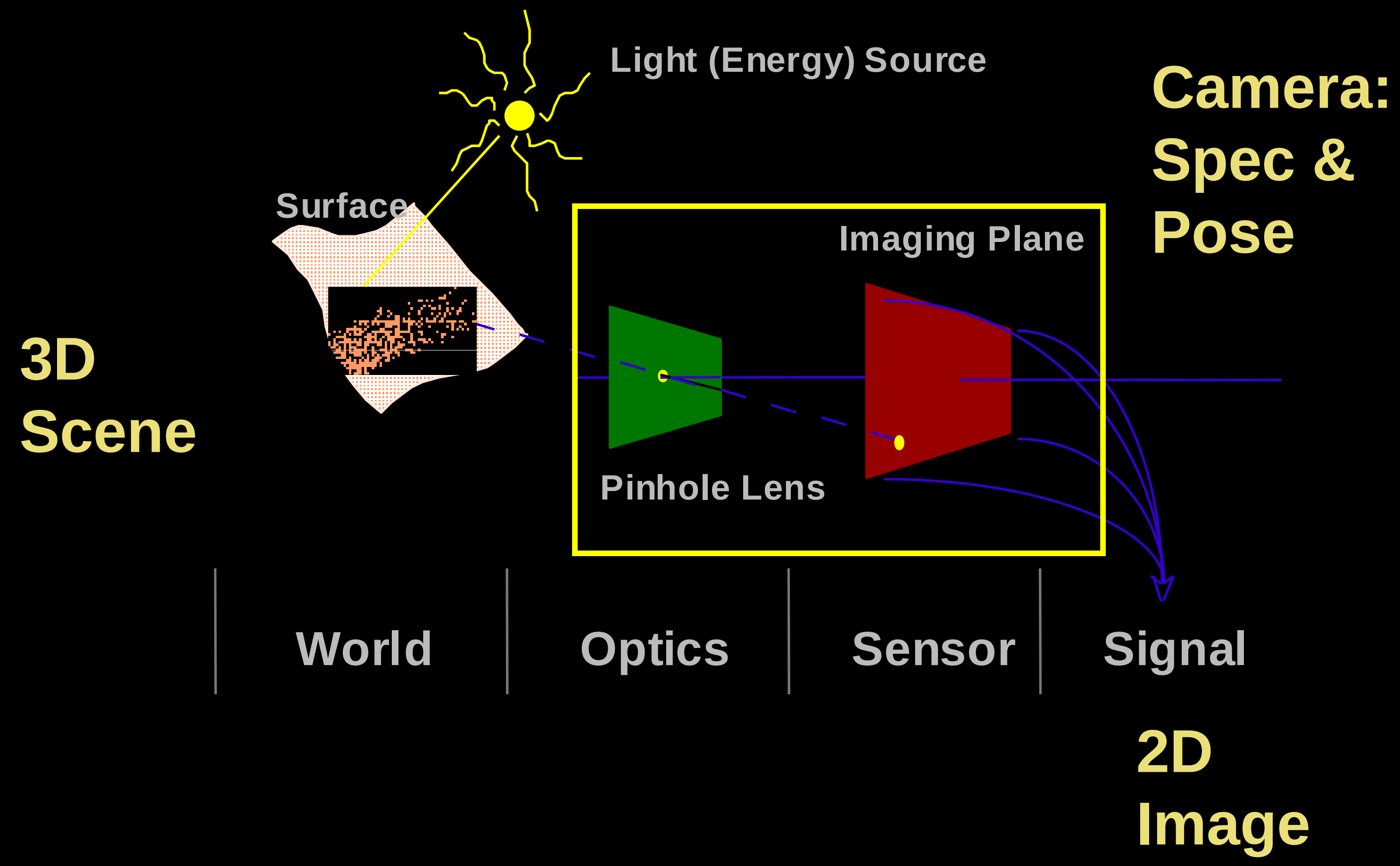
Build up relation between 2D images and 3D scenes
-3D Graphics (rendering): from 3D to 2D
-3D Vision (stereo and motion): from 2D to 3D
-Calibration: Determning the parameters for mapping

Image Formation

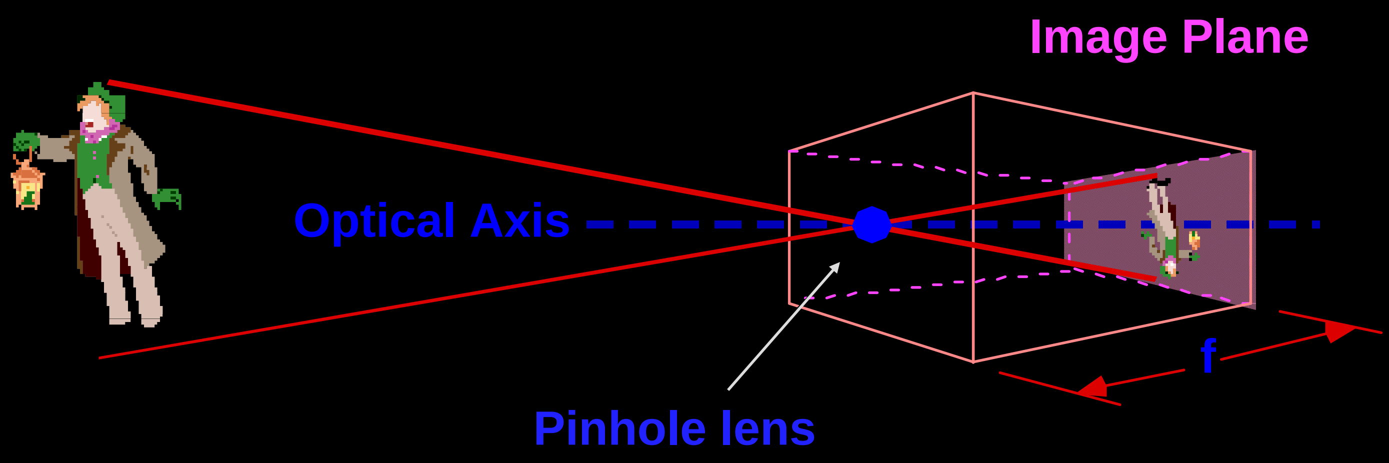


World	Optics	Sensor	Signal
		B&W Film	Silver Density
		Color Film	Silver density in three color layers
		TV Camera	Electrical

Image Formation



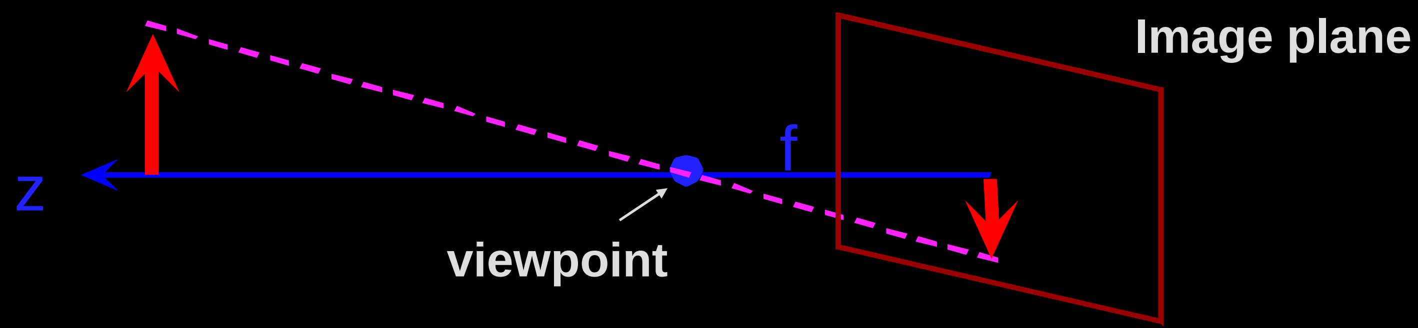
Pinhole Camera Model



- Pin-hole is the basis for most graphics and vision
  - Derived from physical construction of early cameras
  - Mathematics is very straightforward
- 3D World projected to 2D Image
  - Image inverted, size reduced
  - Image is a 2D plane: No direct depth information
- Perspective projection
  - f called the focal length of the lens
  - given image size, change f will change FOV and figure sizes

Focal Length, FOV

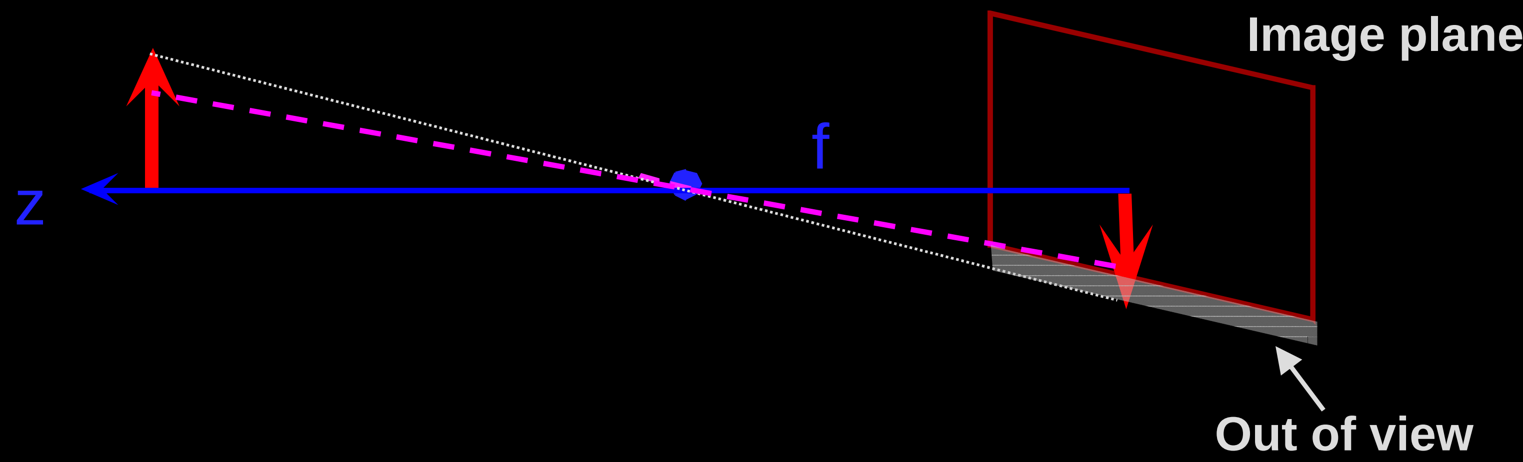
- Consider case with object on the optical axis:



- Optical axis: the direction of imaging
- Image plane: a plane perpendicular to the optical axis
- Center of Projection (pinhole), focal point, viewpoint, nodal point
- Focal length: distance from focal point to the image plane
- FOV : Field of View – viewing angles in horizontal and vertical directions

Focal Length, FOV

- Consider case with object on the optical axis:

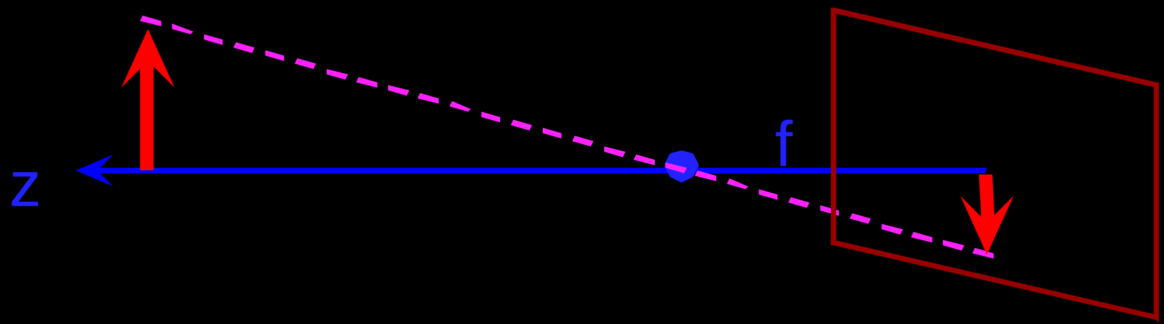


- Optical axis: the direction of imaging
- Image plane: a plane perpendicular to the optical axis
- Center of Projection (pinhole), focal point, viewpoint, , nodal point
- Focal length: distance from focal point to the image plane
- FOV : Field of View – viewing angles in horizontal and vertical directions

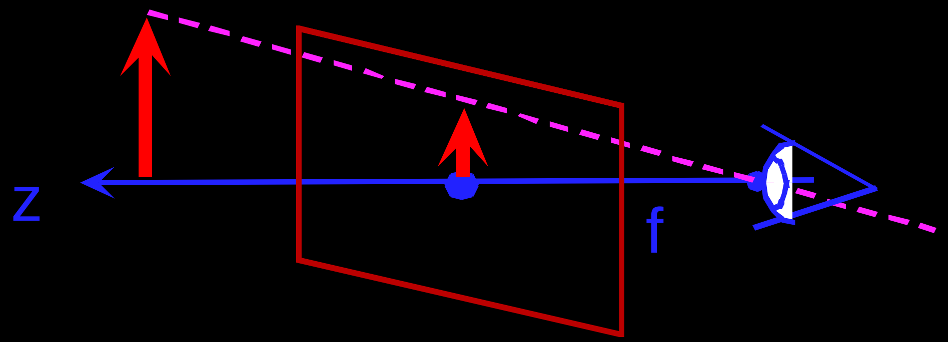
- Increasing f will enlarge figures, but decrease FOV

Equivalent Geometry

- Consider case with object on the optical axis:



- More convenient with upright image:

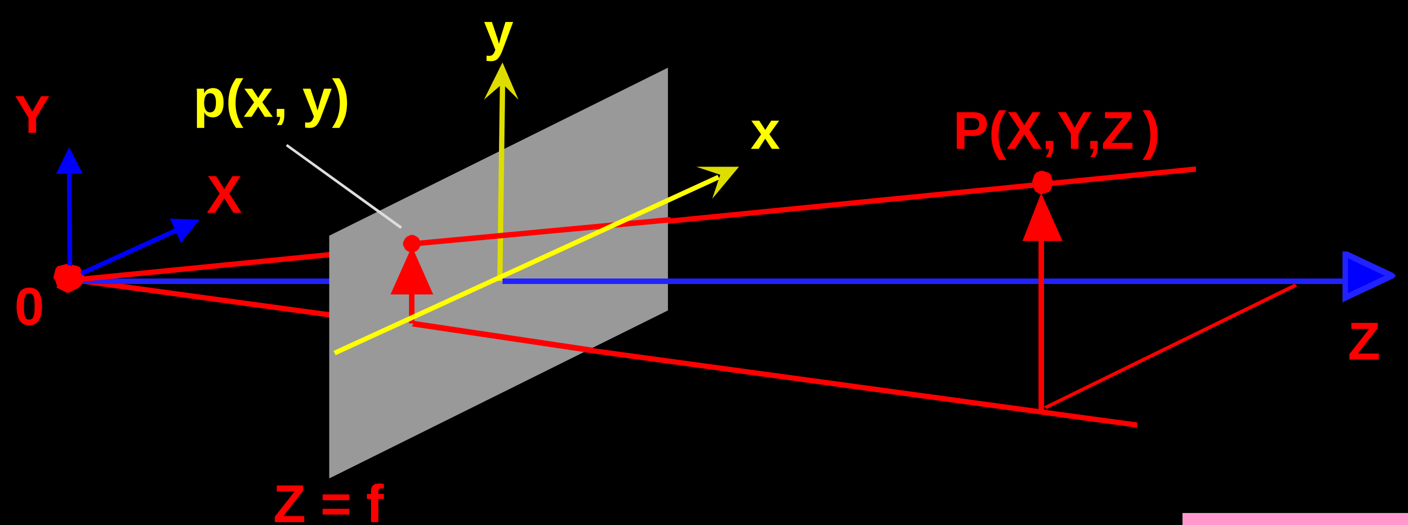


Projection plane z = f

- Equivalent mathematically

Perspective Projection

- Compute the image coordinates of p in terms of the world (camera) coordinates of P.



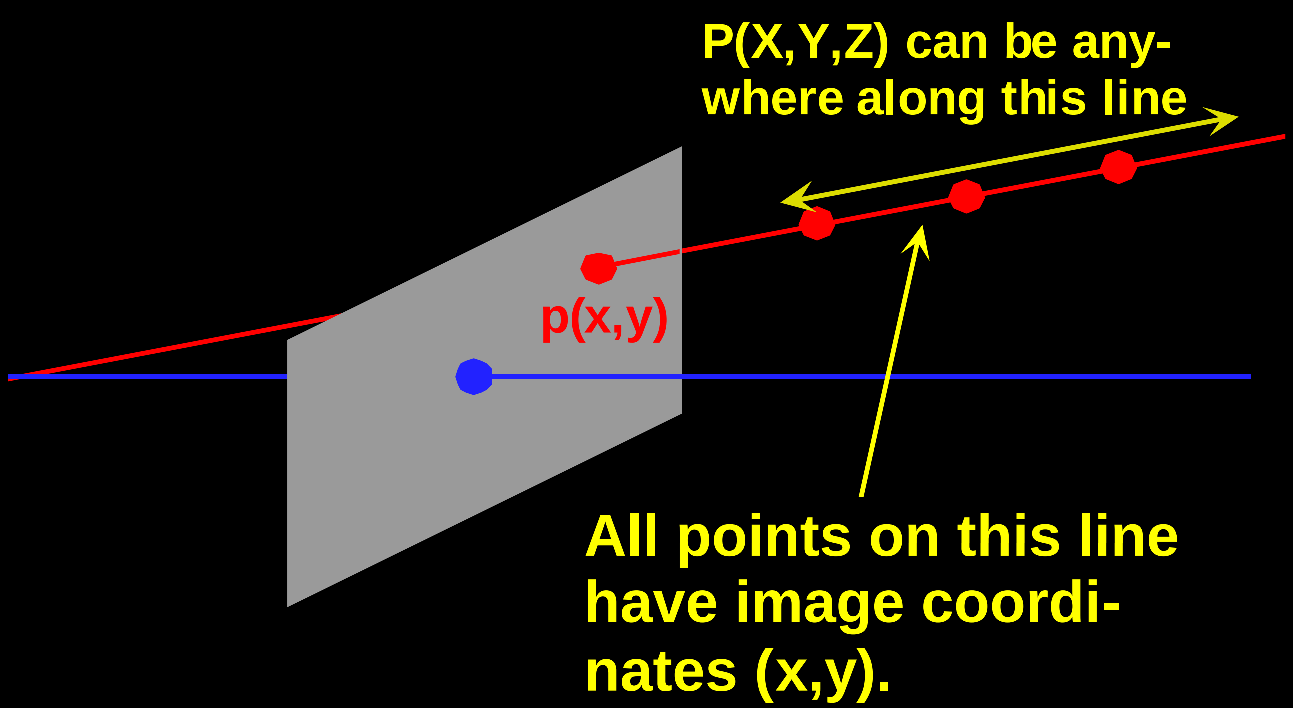
- Origin of camera at center of projection
- Z axis along optical axis
- Image Plane at Z = f; x // X and y//Y

$$x = f\frac{X}{Z}$$

$$y = f\frac{Y}{Z}$$

Reverse Projection

- Given a center of projection and image coordinates of a point, it is not possible to recover the 3D depth of the point from a single image.



In general, at least two images of the same point taken from two different locations are required to recover depth.

Pinhole camera image

Amsterdam : what do you see in this picture?

- straight line
- size
- parallelism/angle
- shape
- shape of planes
- depth

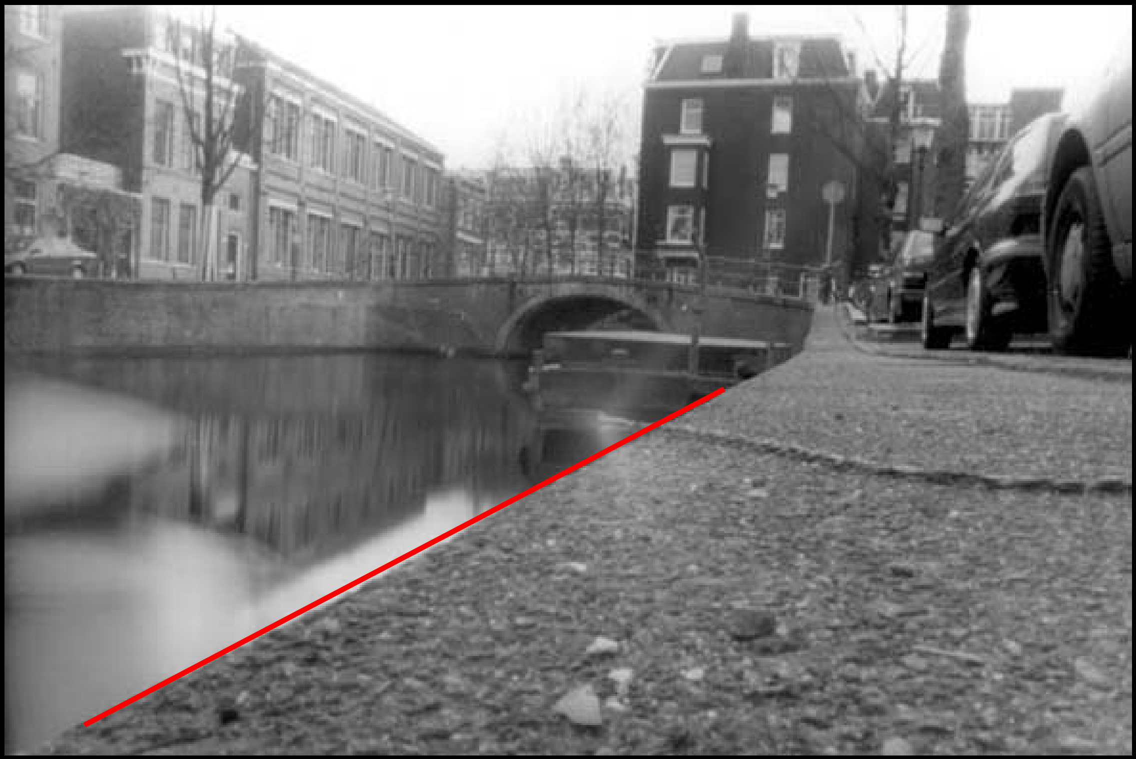
Photo by Robert Kosara, robert@kosara.net
http://www.kosara.net/gallery/pinholeamsterdam/pic01.html

Pinhole camera image

✓straight line

- size
- parallelism/angle
- shape
- shape of planes
- depth

Amsterdam



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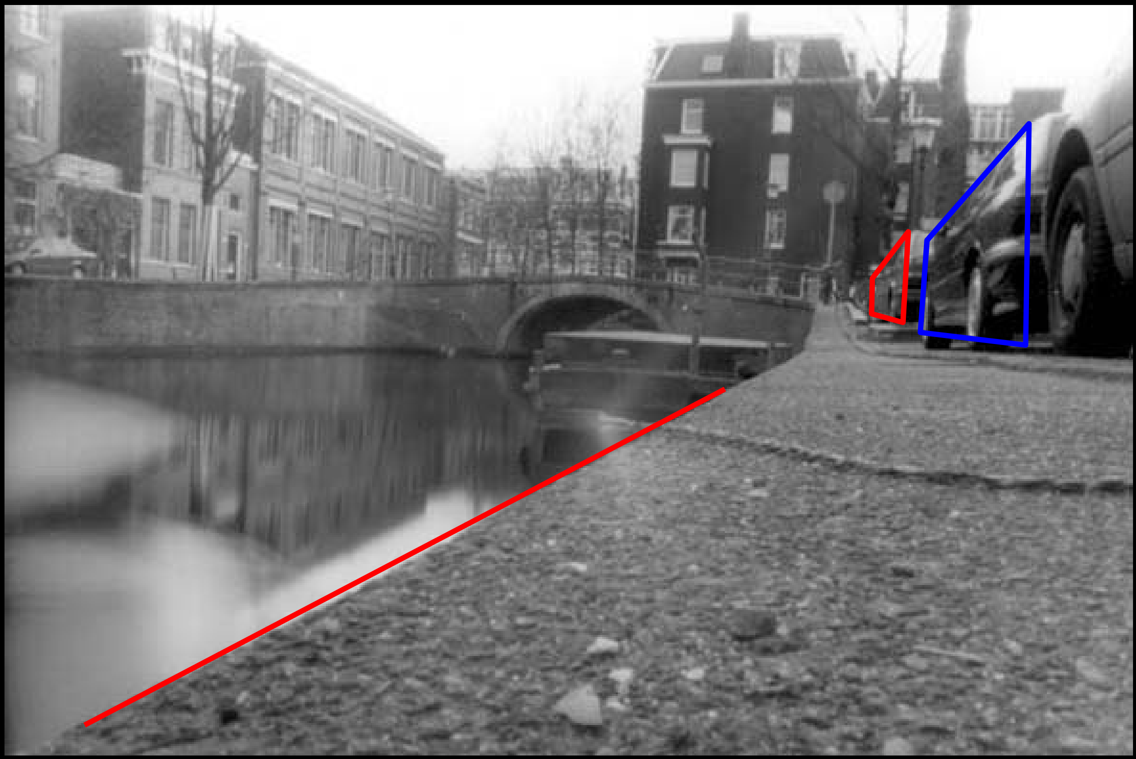
Pinhole camera image

Amsterdam

- ✓straight line
- ×size
- •parallelism/angle
- •shape
- •shape of planes
- •depth



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Pinhole camera image

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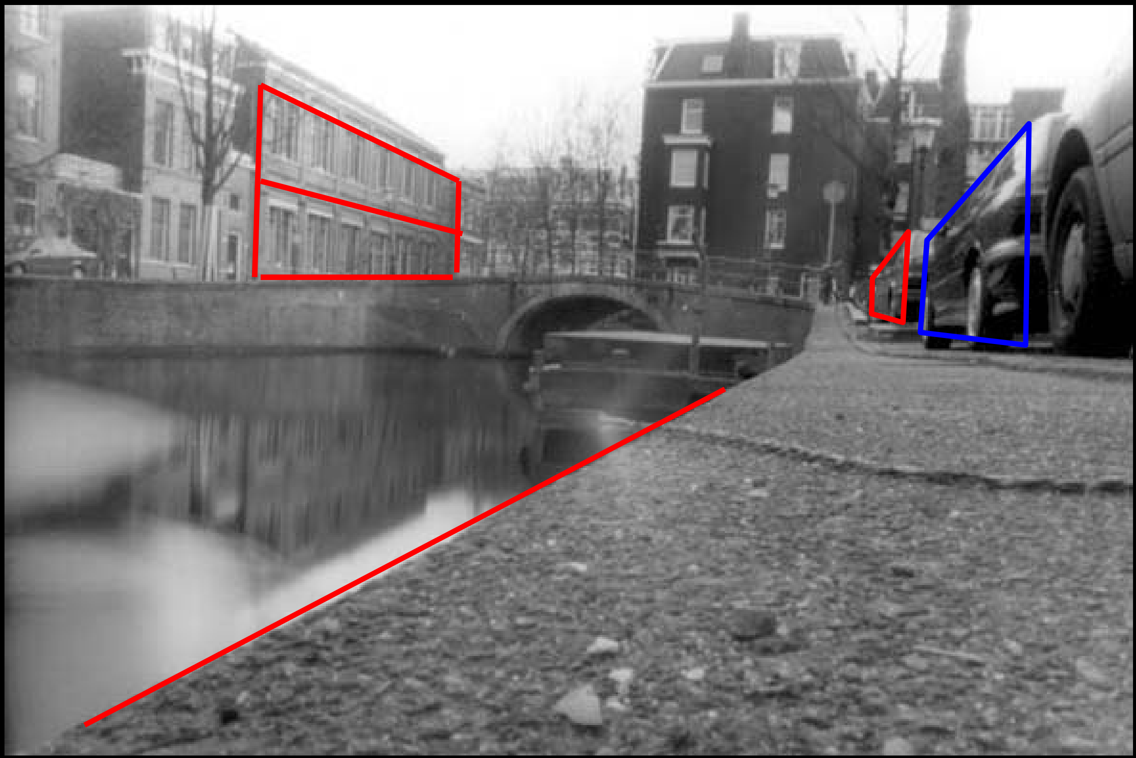

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Pinhole camera image

Amsterdam

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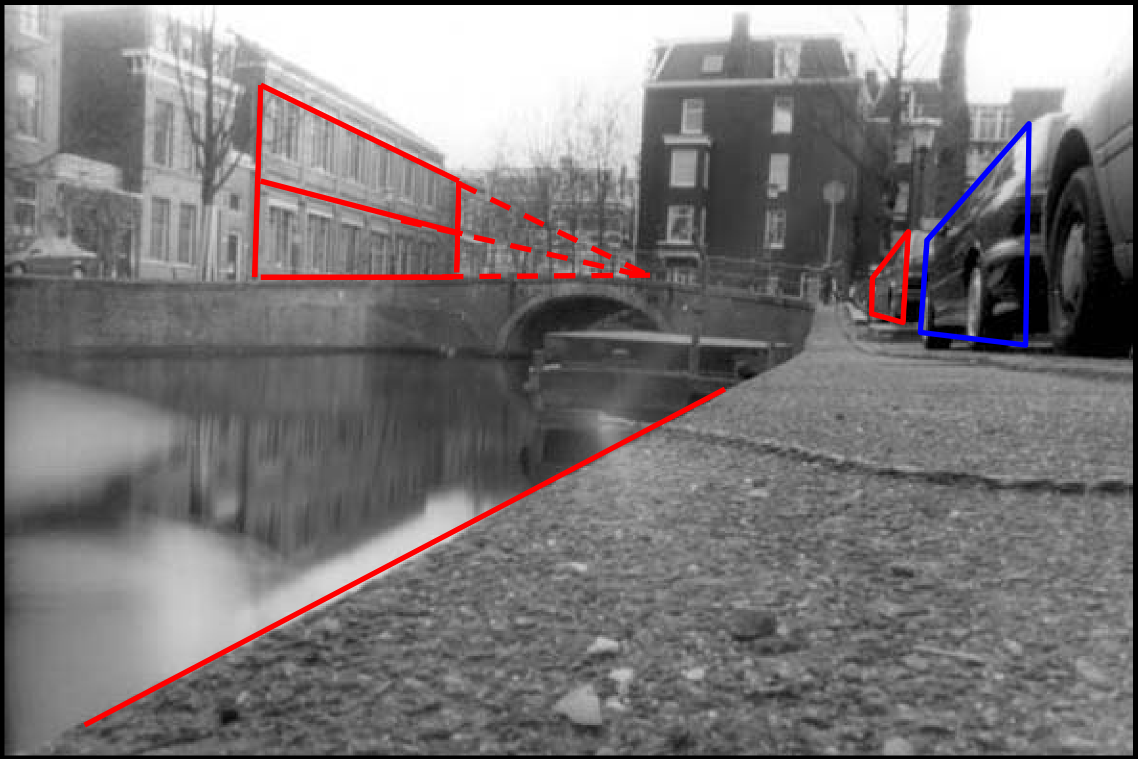


Photo by Robert Kosara, robert@kosara.net
http://www.kosara.net/gallery/pinholeamsterdam/pic01.html

Pinhole camera image

Amsterdam

- ✓straight line
- ×size
- ×parallelism/angle
- ×shape
- ●shape of planes
- ✓ parallel to image
- ●depth

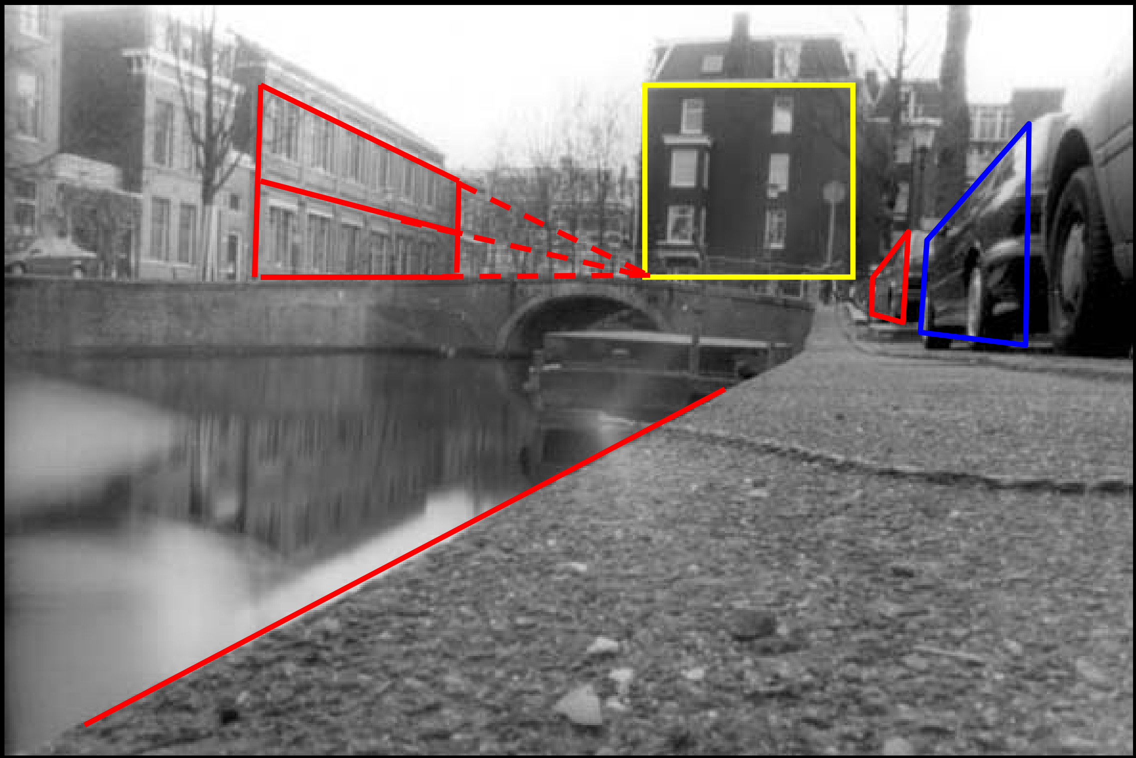


Photo by Robert Kosara, robert@kosara.net
http://www.kosara.net/gallery/pinholeamsterdam/pic01.html

Pinhole camera image

Amsterdam: what do you see?

✓straight line

×size

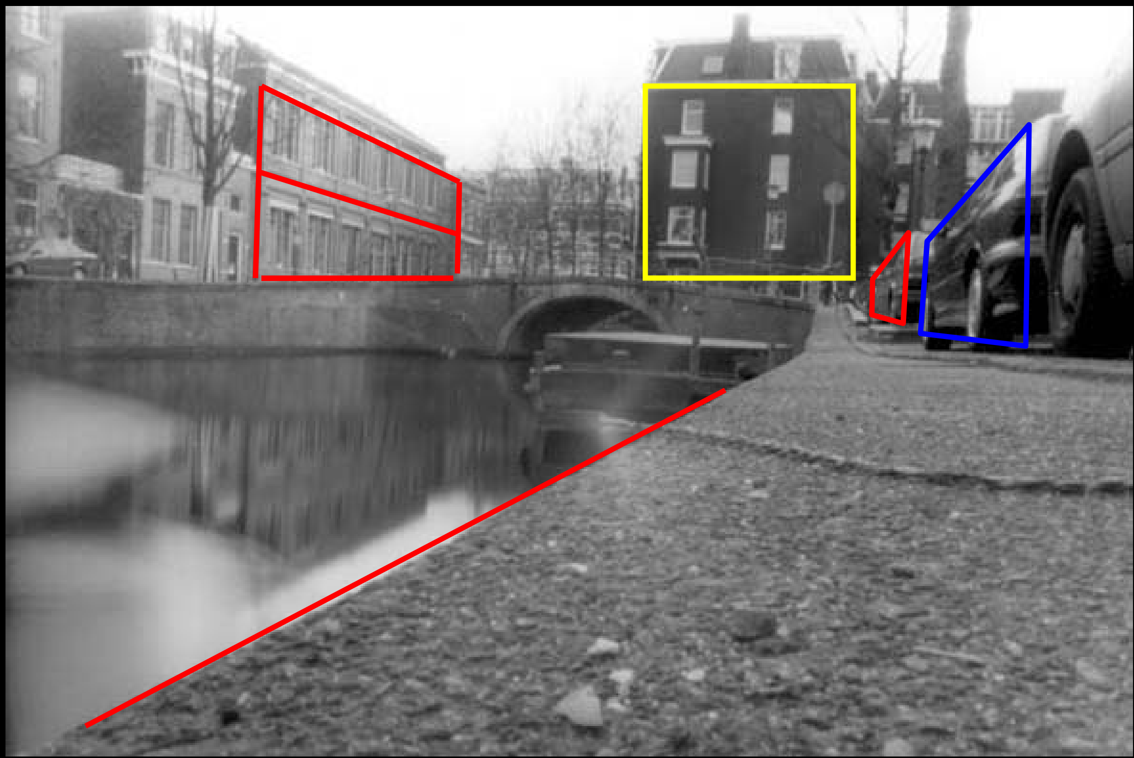
×parallelism/angle

×shape

- shape of planes

✓ parallel to image

- Depth ?
  - stereo
  - motion
  - size
  - structure …



- We see spatial shapes rather than individual pixels
- Knowledge: top-down vision belongs to human
- Stereo &Motion most successful in 3D CV & application
- You can see it but you don't know how…

Yet other pinhole camera images

Rabbit or Man?

Markus Raetz, *Metamorphose II*, 1991-92, cast iron, 15 1/4 x 12 x 12 inches

Fine Art Center University Gallery, Sep 15 – Oct 26

Yet other pinhole camera images

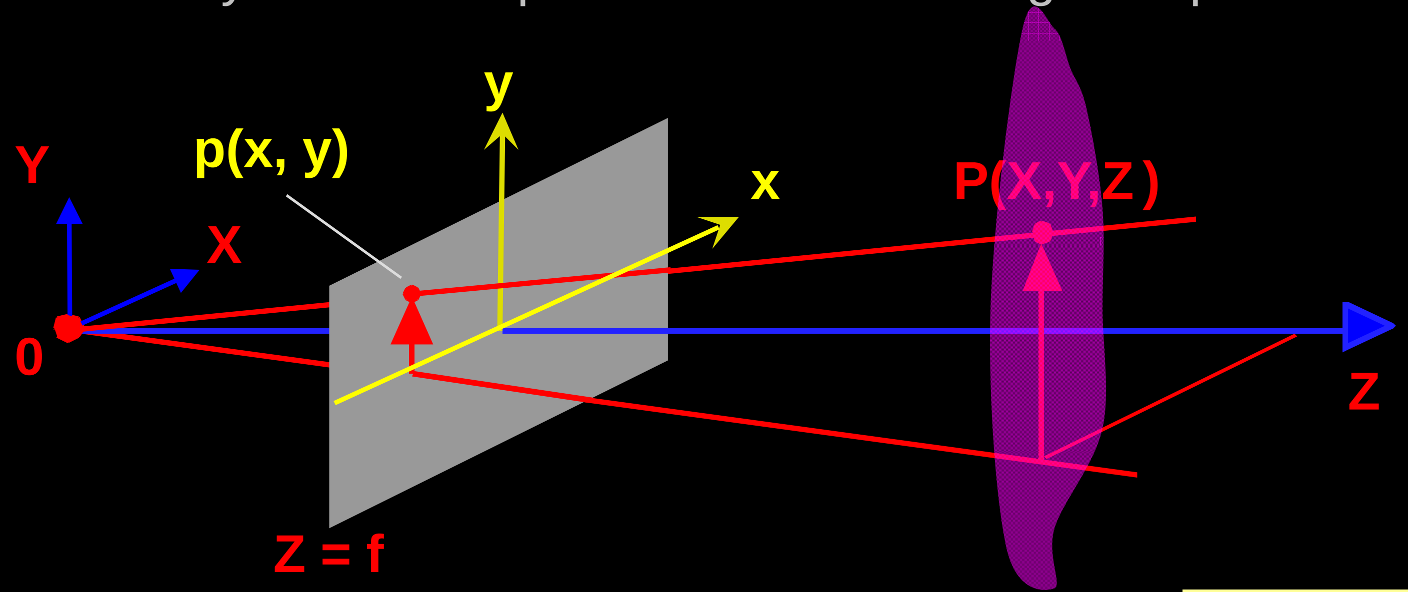
2D projections are not the "same" as the real object as we usually see everyday!

Markus Raetz, *Metamorphose II*, 1991-92, cast iron, 15 1/4 x 12 x 12 inches
Fine Art Center University Gallery, Sep 15 – Oct 26

It’s real!

Weak Perspective Projection

- Average depth $\bar{Z}$ is much larger than the relative distance between any two scene points measured along the optical axis

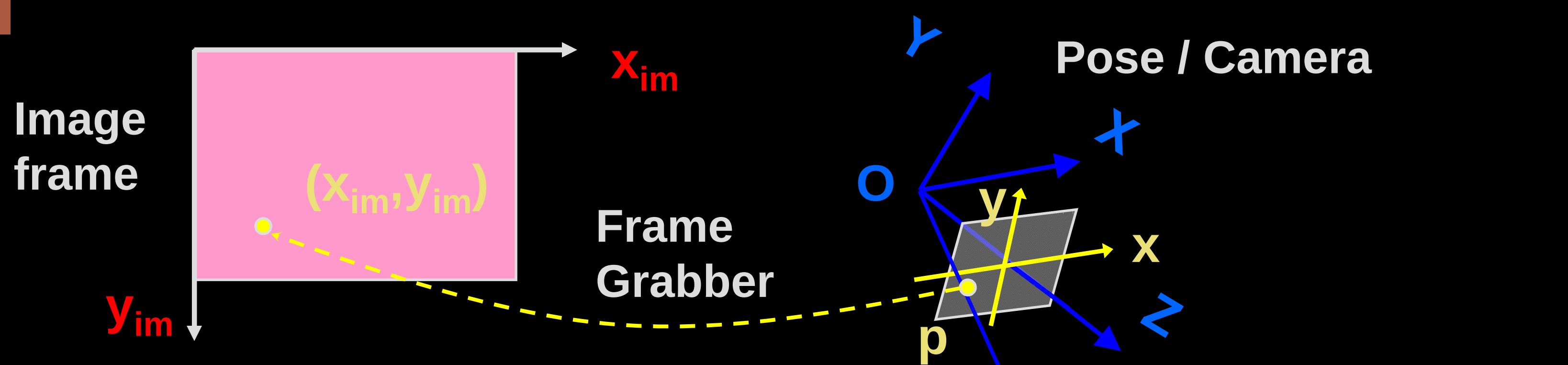


- A sequence of two transformations
  - Orthographic projection : parallel rays
  - Isotropic scaling : $f/\bar{Z}$
- Linear Model
  - Preserve angles and shapes

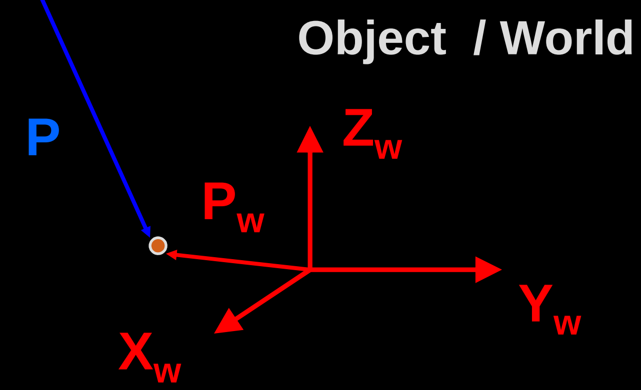
$$x = f\frac{X}{\bar{Z}}$$

$$y = f\frac{Y}{\bar{Z}}$$

Camera Parameters

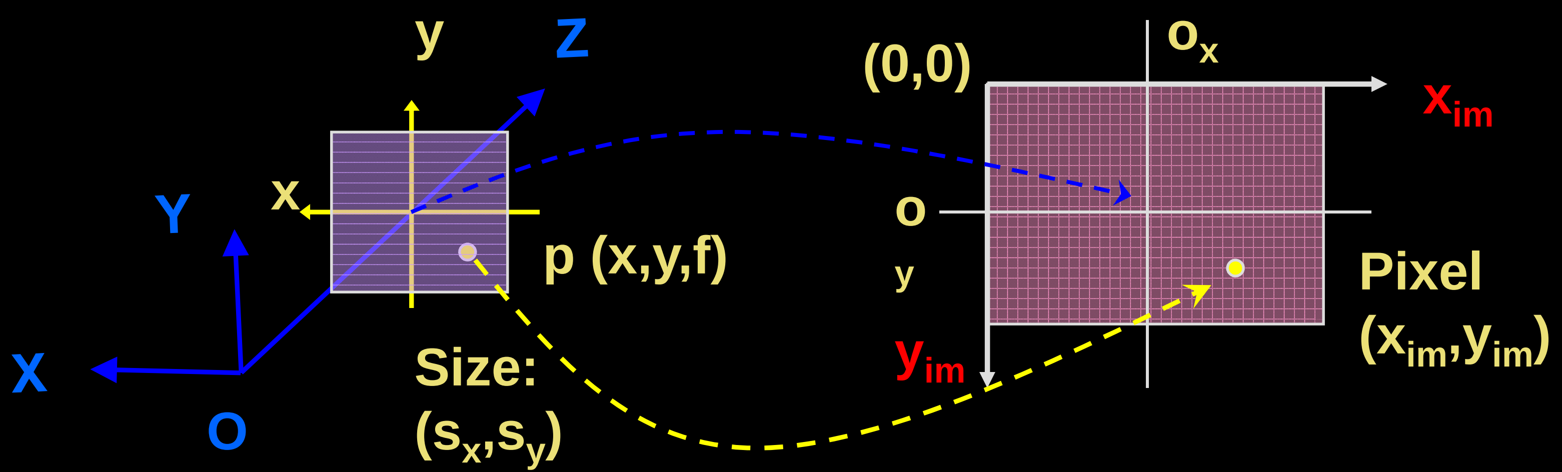


- Coordinate Systems
  - Frame coordinates (x_{im}, y_{im}) pixels
  - Image coordinates (x,y) in mm
  - Camera coordinates (X,Y,Z)
  - World coordinates (X_w,Y_w,Z_w)



- Camera Parameters
  - Intrinsic Parameters (of the camera and the frame grabber): link the **frame coordinates** of an image point with its corresponding **camera coordinates**
  - Extrinsic parameters: define the location and orientation of the **camera coordinate system** with respect to the **world coordinate system**

Intrinsic Parameters (I)



- From image to frame
  - Image center
  - Directions of axes
  - Pixel size
- From 3D to 2D
  - Perspective projection
- Intrinsic Parameters
  - (ox ,oy) : image center (in pixels)
  - (sx ,sy) : effective size of the pixel (in mm)
  - f: focal length

$$x = -(x_{im} - o_x)s_x$$

$$y = -(y_{im} - o_y)s_y$$

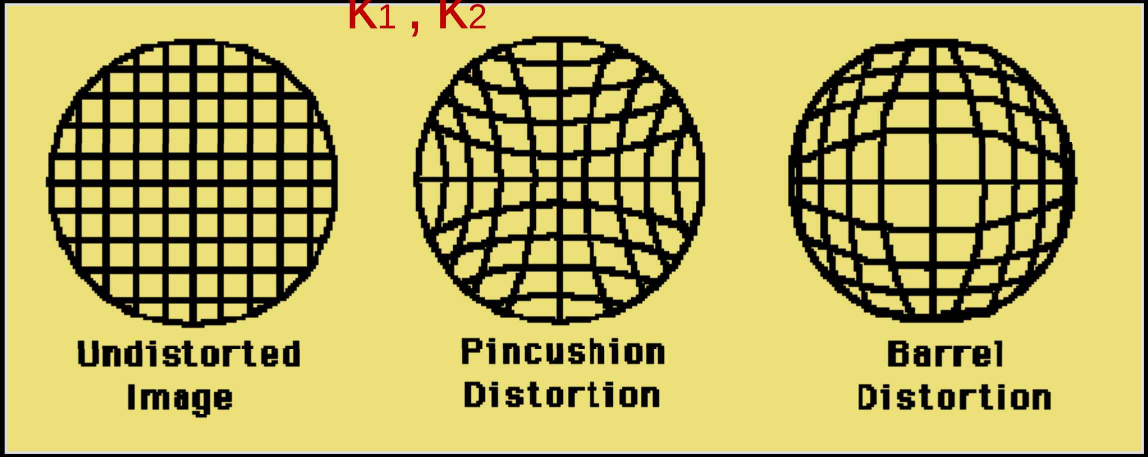
$$x = f\frac{X}{Z}$$

$$y = f\frac{Y}{Z}$$

Intrinsic Parameters (II)

(x, y) ← (xd, yd)
k_1 , k_2

- Lens Distortions

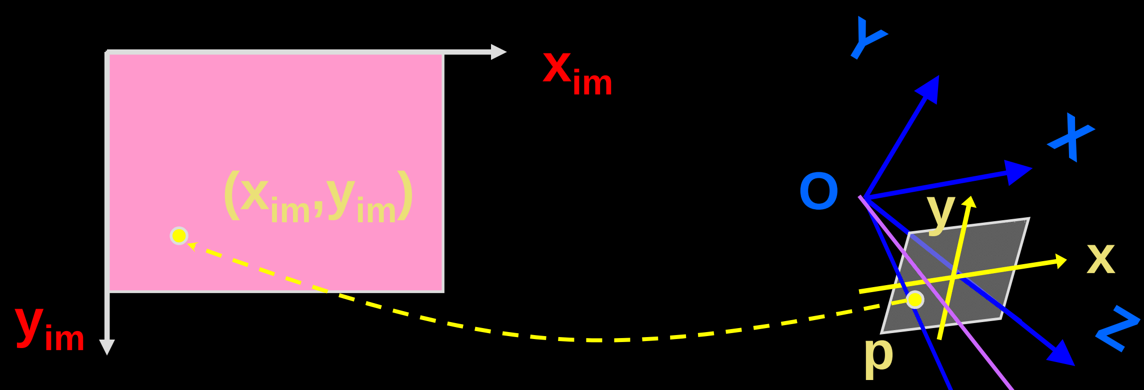


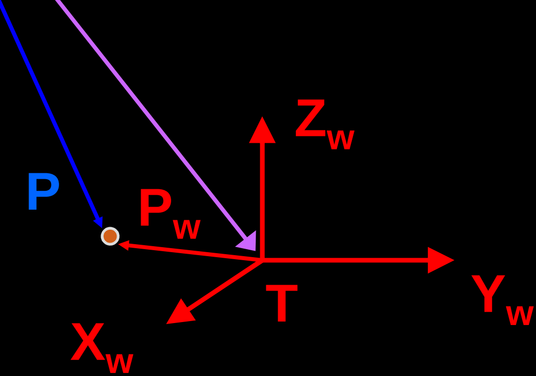
- Modeled as simple radial distortions
  - $r^2 = x_d^2 + y_d^2$
  - (x_d , y_d) distorted points
  - k_1 , k_2: distortion coefficients
  - A model with $k_2 = 0$ is still accurate for a CCD sensor of 500x500 with ~5 pixels distortion on the outer boundary

$$x = x_d (1 + k_1 r^2 + k_2 r^4)$$

$$y = y_d (1 + k_1 r^2 + k_2 r^4)$$

Extrinsic Parameters





- From World to Camera

$$\mathbf{P} = \mathbf{R}\,\mathbf{P_w} + \mathbf{T}$$

- Extrinsic Parameters
  - A 3-D translation vector, T, describing the relative locations of the origins of the two coordinate systems (what's it?)
  - A 3x3 rotation matrix, R, an orthogonal matrix that brings the corresponding axes of the two systems onto each other

Linear Algebra: Vector and Matrix

- A point as a 2D/ 3D vector
  - Image point: 2D vector → $\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix} = (x, y)^T$
  - Scene point: 3D vector → $\mathbf{P} = (X, Y, Z)^T$
  - Translation: 3D vector → $\mathbf{T} = (T_x, T_y, T_z)^T$

T: Transpose

- Vector Operations
  - Addition:
    - Translation of a 3D vector
    - $\mathbf{P} = \mathbf{Pw} + \mathbf{T} = (X_w + T_x, Y_w + T_y, Z_w + T_z)^T$
  - Dot product (a scalar):
    - a.b = |a||b|cosθ
    - $c = \mathbf{a} \bullet \mathbf{b} = \mathbf{a}^T \mathbf{b}$
  - Cross product (a vector)
    - Generates a new vector that is orthogonal to both of them
    - $\mathbf{c} = \mathbf{a} \times \mathbf{b}$

$$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = (a_2 b_3 - a_3 b_2)\underline{\mathbf{i}} + (a_3 b_1 - a_1 b_3)\underline{\mathbf{j}} + (a_1 b_2 - a_2 b_1)\underline{\mathbf{k}}$$

Linear Algebra: Vector and Matrix

- Rotation: 3x3 matrix
  - Orthogonal :

$$\mathbf{R}^{-1} = \mathbf{R}^T, i.e.\ \mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}$$

$$\mathbf{R} = (r_{ij})_{3\times 3} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_1^T \\ \mathbf{R}_2^T \\ \mathbf{R}_3^T \end{bmatrix}$$

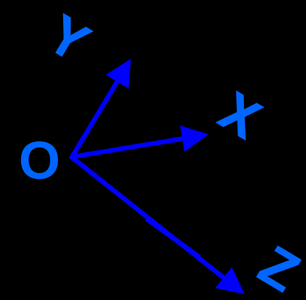
    - 9 elements => 3+3 constraints (orthogonal/cross) => 2+2 constraints (unit vectors) => 3 DOF ? (degrees of freedom, orthogonal/dot)
  - How to generate R from three angles? (next few slides)

- Matrix Operations
  - R P_w +T= ? - Points in the World are projected on three new axes (of the camera system) and translated to a new origin

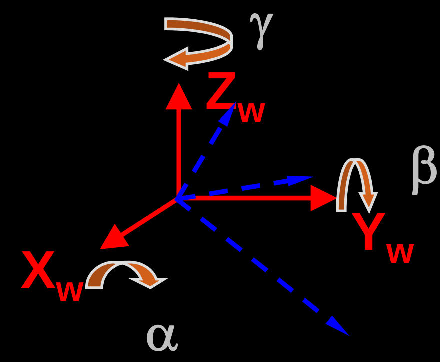
$$\mathbf{P} = \mathbf{R}\mathbf{P_w} + \mathbf{T} = \begin{pmatrix} r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x \\ r_{21}X_w + r_{22}Y_w + r_{23}Z_w + T_y \\ r_{31}X_w + r_{32}Y_w + r_{33}Z_w + T_z \end{pmatrix} = \begin{bmatrix} \mathbf{R}_1^T\mathbf{P_w} + T_x \\ \mathbf{R}_2^T\mathbf{P_w} + T_y \\ \mathbf{R}_3^T\mathbf{P_w} + T_z \end{bmatrix}$$

Rotation: from Angles to Matrix

- Rotation around the Axes
  - Result of three consecutive rotations around the coordinate axes



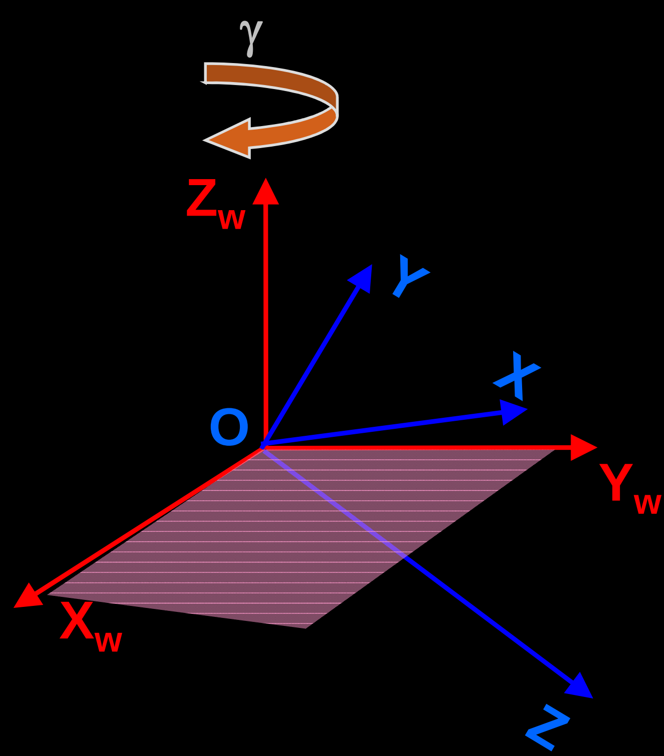
$$\mathbf{R} = \mathbf{R}_{\alpha}\mathbf{R}_{\beta}\mathbf{R}_{\gamma}$$



- Notes:
  - Only three rotations
  - Every time around one axis
  - Bring corresponding axes to each other
    - Xw = X, Yw = Y, Zw = Z
  - First step (e.g.) Bring Xw to X

Rotation: from Angles to Matrix

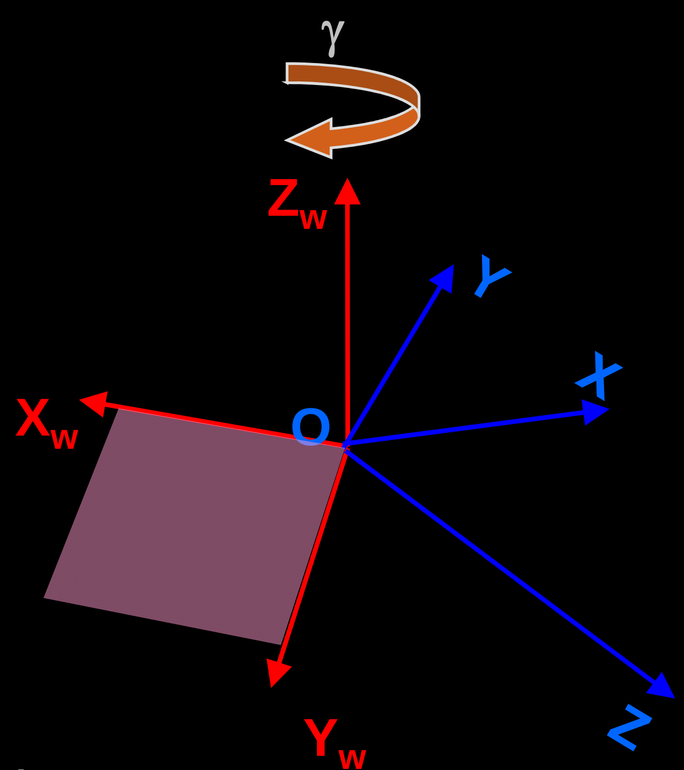
$$\mathbf{R}_\gamma = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



- Rotation γ around the Z_w Axis
  - Rotate in X_wOY_w plane
  - Goal: Bring X_w to X
  - But X is not in XwOYw
  - $Y_w \perp X \Rightarrow X$ in X_wOZ_w ($\Leftarrow Y_w \perp X_wOZ_w$) $\Rightarrow Y_w$ in YOZ ($\Leftarrow X \perp$ YOZ)
- Next time rotation around Y_w

Rotation: from Angles to Matrix

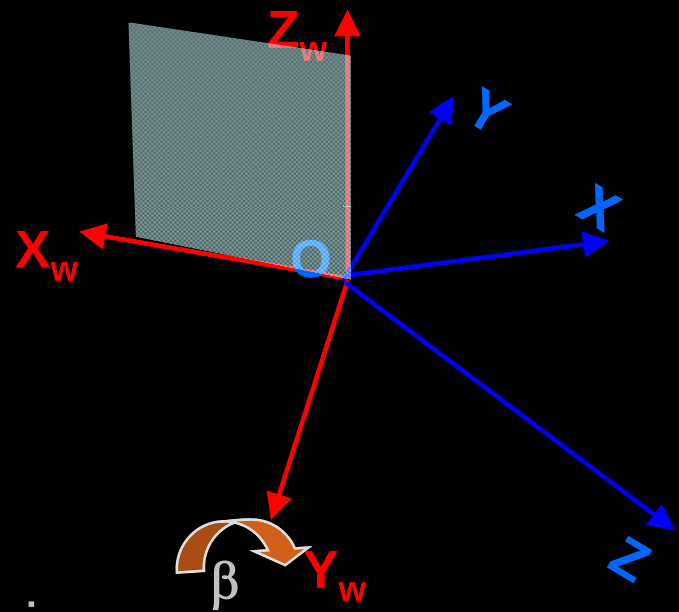
$$\mathbf{R}_\gamma = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



- Rotation γ around the Z_w Axis
  - Rotate in X_wOY_w plane so that
  - $Y_w \perp X \Rightarrow X$ in X_wOZ_w ($\Leftarrow Y_w \perp X_wOZ_w$) $\Rightarrow Y_w$ in YOZ ($\Leftarrow X \perp YOZ$)
- Z_w does not change

Rotation: from Angles to Matrix

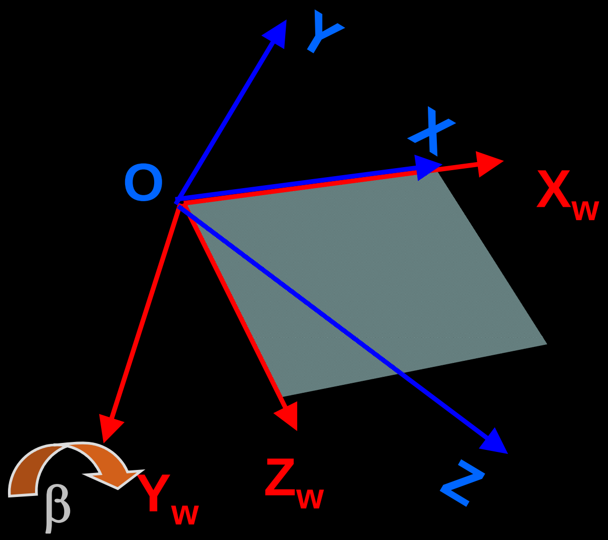
$$\mathbf{R}_{\gamma} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$$



- Rotation β around the Y_w Axis
  - Rotate in X_wOZ_w plane so that
  - $X_w = X \Rightarrow Z_w$ in YOZ (& Y_w in YOZ)
- Y_w does not change

Rotation: from Angles to Matrix

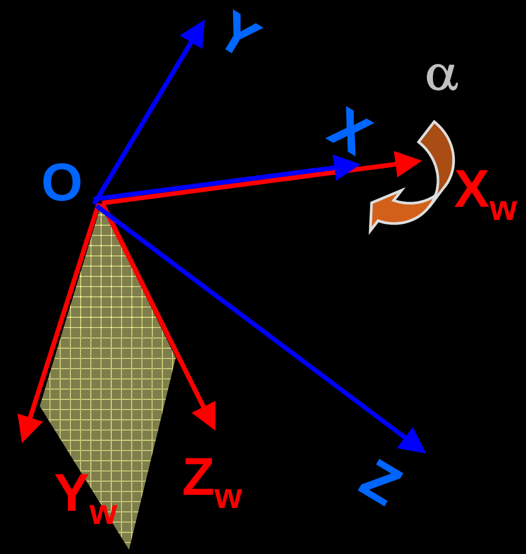
$$\mathbf{R}_\beta = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$$



- Rotation β around the Y_w Axis
  - Rotate in X_wOZ_w plane so that
  - $X_w = X \Rightarrow Z_w$ in YOZ (& Y_w in YOZ)
- Y_w does not change

Rotation: from Angles to Matrix

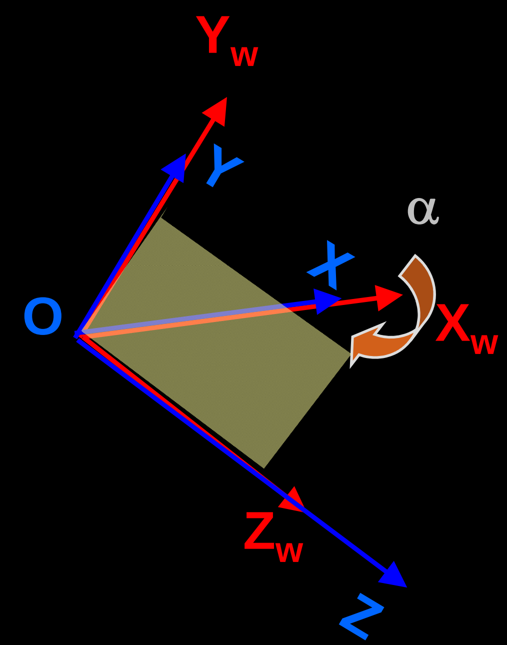
$$\mathbf{R}_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$



- Rotation α around the X_w(X) Axis
  - Rotate in Y_wOZ_w plane so that
  - $Y_w = Y$, $Z_w = Z$ (& $X_w = X$)
- X_w does not change

Rotation: from Angles to Matrix

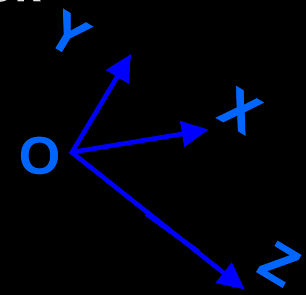
$$\mathbf{R}_{\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$



- Rotation α around the X_w(X) Axis
  - Rotate in Y_wOZ_w plane so that
  - $Y_w = Y$, $Z_w = Z$ (& $X_w = X$)
- X_w does not change

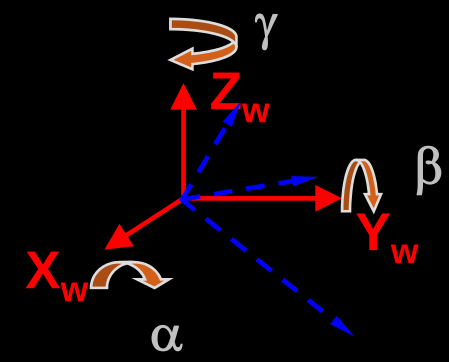
Rotation: from Angles to Matrix

Appendix A.9 of the textbook



- Rotation around the Axes
  - Result of three consecutive rotations around the coordinate axes

$$\mathbf{R} = \mathbf{R}_{\alpha}\mathbf{R}_{\beta}\mathbf{R}_{\gamma}$$



- Notes:
  - Rotation directions
  - The order of multiplications matters: γ,β,α
  - Same R, 6 different sets of α,β,γ
  - **R Non-linear function of** α,β,γ
  - **R is orthogonal**
  - **It's easy to compute angles from R**

$$\mathbf{R} = \begin{bmatrix} \cos\beta\cos\gamma & -\cos\beta\sin\gamma & -\sin\beta \\ -\sin\alpha\sin\beta\cos\gamma + \cos\alpha\sin\gamma & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & -\sin\alpha\cos\beta \\ \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma & -\cos\alpha\sin\beta\sin\gamma + \sin\alpha\cos\gamma & \cos\alpha\cos\gamma \end{bmatrix}$$

Rotation- Axis and Angle

Appendix A.9 of the textbook

- According to Euler's Theorem, any 3D rotation can be described by a rotating angle, θ, around an axis defined by an unit vector **n** = $[n_1, n_2, n_3]^T$.
- Three degrees of freedom – why?

$$\mathbf{R} = I\cos\theta + \begin{bmatrix} n_1^2 & n_1 n_2 & n_1 n_3 \\ n_2 n_1 & n_2^2 & n_2 n_3 \\ n_3 n_1 & n_3 n_2 & n_3^2 \end{bmatrix}(1-\cos\theta) + \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}\sin\theta$$

Linear Version of Perspective Projection

- World to Camera
  - Camera: P = $(X,Y,Z)^T$
  - World: Pw = $(Xw,Yw,Zw)^T$
  - Transform: R, T
- Camera to Image
  - Camera: P = $(X,Y,Z)^T$
  - Image: p = $(x,y)^T$
  - Not linear equations
- Image to Frame
  - Neglecting distortion
  - Frame $(xim, yim)^T$
- World to Frame
  - $(Xw,Yw,Zw)^T$ -> $(xim, yim)^T$
  - Effective focal lengths
    - $f_x = f/s_x$, $f_y=f/s_y$
    - Three are not independent

$$\mathbf{P} = \mathbf{R}\mathbf{P_w} + \mathbf{T} = \begin{pmatrix} r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x \\ r_{21}X_w + r_{22}Y_w + r_{23}Z_w + T_y \\ r_{31}X_w + r_{32}Y_w + r_{33}Z_w + T_z \end{pmatrix} = \begin{bmatrix} \mathbf{R}_1^T\mathbf{P_w} + T_x \\ \mathbf{R}_2^T\mathbf{P_w} + T_y \\ \mathbf{R}_3^T\mathbf{P_w} + T_z \end{bmatrix}$$

$$(x, y) = (f\frac{X}{Z},\ \ f\frac{Y}{Z})$$

$$x = -(x_{im} - o_x)s_x$$

$$y = -(y_{im} - o_y)s_y$$

$$x_{im} - o_x = -f_x \frac{r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x}{r_{31}X_w + r_{32}Y_w + r_{33}Z_w + T_z}$$

$$y_{im} - o_y = -f_y \frac{r_{21}X_w + r_{22}Y_w + r_{23}Z_w + T_y}{r_{31}X_w + r_{32}Y_w + r_{33}Z_w + T_z}$$

Linear Matrix Equation of perspective projection

- Projective Space
  - Add fourth coordinate
    - $P_w = (X_w, Y_w, Z_w, 1)^T$
  - Define $(x1,x2,x3)^T$ such that
    - $x_1/x_3 = x_{im}$, $x_2/x_3 = y_{im}$
- 3x4 Matrix M_{ext}
  - Only extrinsic parameters
  - World to camera
- 3x3 Matrix M_{int}
  - Only intrinsic parameters
  - Camera to frame
- Simple Matrix Product! Projective Matrix $M = M_{int}M_{ext}$
  - $(X_w, Y_w, Z_w)^T \rightarrow (x_{im}, y_{im})^T$
  - Linear Transform from projective space to projective plane
  - M defined up to a scale factor – 11 independent entries

$$\begin{pmatrix} x_{im} \\ y_{im} \end{pmatrix} = \begin{pmatrix} x_1/x_3 \\ x_2/x_3 \end{pmatrix} \Leftarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{M}_{\mathbf{int}}\mathbf{M}_{\mathbf{ext}} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}$$

$$\mathbf{M}_{ext} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & T_x \\ r_{21} & r_{22} & r_{23} & T_y \\ r_{31} & r_{32} & r_{33} & T_z \end{bmatrix} = \begin{bmatrix} \mathbf{R}_1^T & T_x \\ \mathbf{R}_2^T & T_y \\ \mathbf{R}_3^T & T_z \end{bmatrix}$$

$$\mathbf{M}_{\text{int}} = \begin{bmatrix} -f_x & 0 & o_x \\ 0 & -f_y & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

Three Camera Models

- Perspective Camera Model
  - Making some assumptions
    - Known center: Ox = Oy = 0
    - Square pixel: Sx = Sy = 1
  - 11 independent entries <-> 7 parameters

$$\mathbf{M} = \begin{bmatrix} -fr_{11} & -fr_{12} & -fr_{13} & -fT_x \\ -fr_{21} & -fr_{22} & -fr_{23} & -fT_y \\ r_{31} & r_{32} & r_{33} & T_z \end{bmatrix}$$

- Weak-Perspective Camera Model
  - Average Distance $\bar{Z}$ >> Range δZ
  - Define centroid vector $\bar{P}_w$

$$Z = \bar{Z} = \mathbf{R}_3^T \bar{\mathbf{P}}_w + T_z$$

  - 8 independent entries

$$\mathbf{M}_{wp} = \begin{bmatrix} -fr_{11} & -fr_{12} & -fr_{13} & -fT_x \\ -fr_{21} & -fr_{22} & -fr_{23} & -fT_y \\ 0 & 0 & 0 & \mathbf{R}_3^T \bar{\mathbf{P}}_w + T_z \end{bmatrix}$$

- Affine Camera Model
  - Mathematical Generalization of Weak-Pers
  - Doesn't correspond to physical camera
  - But simple equation and appealing geometry
    - Doesn't preserve angle BUT parallelism
  - 8 independent entries

$$\mathbf{M}_{af} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ 0 & 0 & 0 & b_3 \end{bmatrix}$$

Camera Models for a Plane

- Planes are very common in the Man-Made World

$$n_x X_w + n_y Y_w + n_z Z_w = d \quad \Longleftrightarrow \quad \mathbf{n}^T \mathbf{P_w} = d$$

  - One more constraint for all points: Zw is a function of Xw and Yw
- Special case: Ground Plane
  - Zw=0
  - Pw =(Xw, Yw,0, 1)^T
  - 3D point -> 2D point

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} -fr_{11} & -fr_{12} & -fr_{13} & -fT_x \\ -fr_{21} & -fr_{22} & -fr_{23} & -fT_y \\ r_{31} & r_{32} & r_{33} & T_z \end{bmatrix} \begin{pmatrix} X_w \\ Y_w \\ Z_w = 0 \\ 1 \end{pmatrix}$$

- Projective Model of a Plane
  - 8 independent entries
- General Form ?
  - 8 independent entries

Camera Models for a Plane

- A Plane in the World

$$n_x X_w + n_y Y_w + n_z Z_w = d \quad \Longleftrightarrow \quad \mathbf{n}^{\mathbf{T}}\mathbf{P}_{\mathbf{w}} = d$$

  - One more constraint for all points: Zw is a function of Xw and Yw
- Special case: Ground Plane
  - $Z_w=0$
  - $P_w = (X_w, Y_w, 0, 1)^T$
  - 3D point -> 2D point

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} -fr_{11} & -fr_{12} & -fr_{13} & -fT_x \\ -fr_{21} & -fr_{22} & -fr_{23} & -fT_y \\ r_{31} & r_{32} & r_{33} & T_z \end{bmatrix} \begin{pmatrix} X_w \\ Y_w \\ Z_w = 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} -fr_{11} & -fr_{12} & -fT_x \\ -fr_{21} & -fr_{22} & -fT_{23} \\ r_{31} & r_{32} & T_z \end{bmatrix} \begin{pmatrix} X_w \\ Y_w \\ 1 \end{pmatrix}$$

- Projective Model of $Z_w=0$
  - 8 independent entries
- General Form ?
  - 8 independent entries

Camera Models for a Plane

- A Plane in the World

$n_x X_w + n_y Y_w + n_z Z_w = d$ ⟺ $\mathbf{n}^T \mathbf{P_w} = d$

  - One more constraint for all points: Zw is a function of Xw and Yw

- Special case: Ground Plane
  - Zw=0
  - Pw =(Xw, Yw,0, 1)^T
  - 3D point -> 2D point

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} -fr_{11} & -fr_{12} & -fr_{13} & -fT_x \\ -fr_{21} & -fr_{22} & -fr_{23} & -fT_y \\ r_{31} & r_{32} & r_{33} & T_z \end{bmatrix} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}$$

- Projective Model of Zw=0
  - 8 independent entries

- General Form ?
  - nz = 1

$Z_w = d - n_x X_w - n_y Y_w$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} -f(r_{11} - n_x r_{13}) & -f(r_{12} - n_y r_{13}) & -f(dr_{13} + T_x) \\ -f(r_{21} - n_x r_{23}) & -f(r_{22} - n_y r_{23}) & -f(dr_{23} + T_y) \\ (r_{31} - n_x r_{33}) & (r_{32} - n_y r_{33}) & (dr_{33} + T_z) \end{bmatrix} \begin{pmatrix} X_w \\ Y_w \\ 1 \end{pmatrix}$$

  - 8 independent entries

- **2D (x_{im}, y_{im}) -> 3D (X_w, Y_w, Z_w) ?**

Applications and Issues

- Graphics /Rendering
  - From 3D world to 2D image
    - Changing viewpoints and directions
    - Changing focal length
  - Fast rendering algorithms
- Vision / Reconstruction
  - From 2D image to 3D model
    - Inverse problem
    - Much harder / unsolved
  - Robust algorithms for matching and parameter estimation
  - Need to estimate camera parameters first
- Calibration
  - Find intrinsic & extrinsic parameters
  - Given image-world point pairs
  - Probably a partially solved problem ?
  - 11 independent entries
    - <-> 10 parameters: fx, fy, ox, oy, α,β,γ, Tx,Ty,Tz

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{M}_{\mathbf{int}} \mathbf{M}_{\mathbf{ext}} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_{im} \\ y_{im} \end{pmatrix} = \begin{pmatrix} x_1 / x_3 \\ x_2 / x_3 \end{pmatrix}$$

$$\mathbf{M}_{\text{int}} = \begin{bmatrix} -f_x & 0 & o_x \\ 0 & -f_y & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}_{ext} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & T_x \\ r_{21} & r_{22} & r_{23} & T_y \\ r_{31} & r_{32} & r_{33} & T_z \end{bmatrix}$$

Camera Model Summary

- Geometric Projection of a Camera
  - Pinhole camera model
  - Perspective projection
  - Weak-Perspective Projection
- Camera Parameters (10 or 11)
  - Intrinsic Parameters: f, o_x,o_y, s_x,s_y,k_1: 4 or 5 independent parameters
  - Extrinsic parameters: R, T – 6 DOF (degrees of freedom)
- Linear Equations of Camera Models (without distortion)
  - General Projection Transformation Equation : 11 parameters
  - Perspective Camera Model: 11 parameters
  - Weak-Perspective Camera Model: 8 parameters
  - Affine Camera Model: generalization of weak-perspective: 8
  - Projective transformation of planes: 8 parameters

Next

- Determining the value of the extrinsic and intrinsic parameters of a camera

Calibration (Ch. 6)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{M}_{\mathbf{int}}\mathbf{M}_{\mathbf{ext}} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}$$

$$\mathbf{M}_{\text{int}} = \begin{bmatrix} -f_x & 0 & o_x \\ 0 & -f_y & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}_{ext} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & T_x \\ r_{21} & r_{22} & r_{23} & T_y \\ r_{31} & r_{32} & r_{33} & T_z \end{bmatrix}$$