Spatial Transformations

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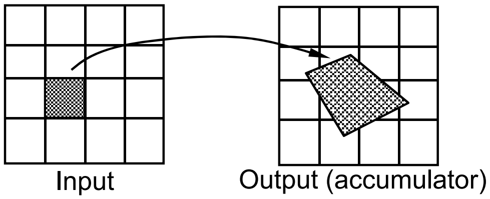
Objectives

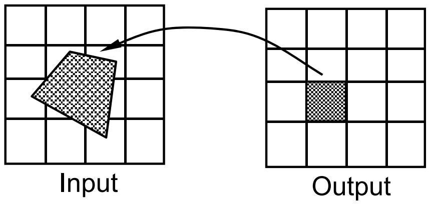
- In this lecture we review spatial transformations:
  - Forward and inverse mappings
  - Transformations
    - Linear
    - Affine
    - Perspective
    - Bilinear
  - Inferring affine and perspective transformations

Forward and Inverse Mappings

- A spatial transformation defines a geometric relationship between each point in the input and output images.
- Forward mapping: $[x, y] = [X(u,v), Y(u,v)]$
- Inverse mapping : $[u, v] = [U(x,y), V(x,y)]$
- Forward mapping specifies output coordinates *(x,y)* for each input point *(u,v)*.
- Inverse mapping specifies input point *(u,v)* for each output point *(x,y).*



Forward mapping



Inverse mapping

Linear Transformations

$$[x, y] = [u, v]\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$x = a_{11}u + a_{21}v$$

$$y = a_{12}u + a_{22}v$$

- Above equations are linear because they satisfy the following two conditions necessary for any linear function L(x):

1) L(x+y) = L(x) + L(y)
2) L(cx) = cL(x) for any scalar c and position vectors x and y.

- Note that linear transformation are a sum of scaled input coordinate: they do not account for simple translation.

We want:

$$x = a_{11}u + a_{21}v + a_{31}$$

$$y = a_{12}u + a_{22}v + a_{32}$$

$$[x, y] = [u, v, 1]\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Homogeneous Coordinates

- To compute inverse, the transformation matrix must be square.
- Therefore,

$$[x, y, 1] = [u, v, 1]\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{bmatrix}$$

- All 2-D position vectors are now represented with three components: homogeneous notation.
- Third component (w) refers to the plane upon which the transformation operates.
- [u, v, 1] = [u, v] position vector lying on w=1 plane.
- The representation of a point in the homogeneous notation is no longer unique: [8, 16, 2] = [4, 8, 1] = [16, 32, 4].
- To recover any 2-D position vector *p[x,y]* from $p_h=[x', y', w']$, divide by the homogeneous coordinate w'.

$$[x, y] = \begin{bmatrix} \frac{x'}{w'} & \frac{y'}{w'} \end{bmatrix}$$

Affine Transformations (1)

$$[x, y, 1] = [u, v, 1] \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{bmatrix}$$

$$\text{Translation} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ T_u & T_v & 1 \end{bmatrix}$$

$$\text{Rotation} \rightarrow \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Scale} \rightarrow \begin{bmatrix} S_u & 0 & 0 \\ 0 & S_v & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Shear rows} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ H_u & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Shear columns} \rightarrow \begin{bmatrix} 1 & H_u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

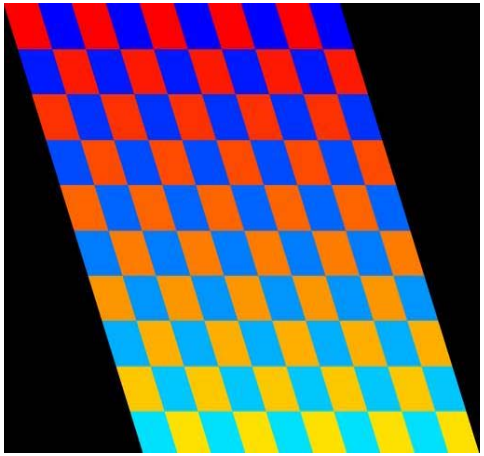
Affine Transformations (2)

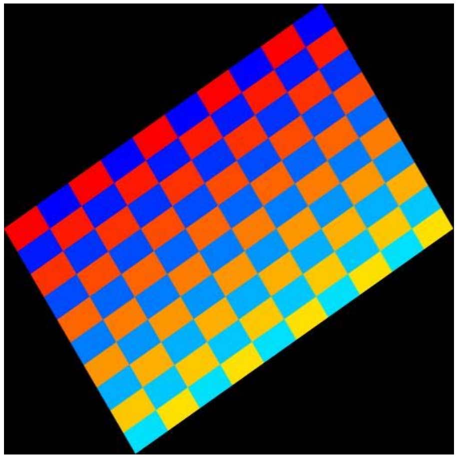
- Affine transformation have 6 degrees of freedom: a_{11}, a_{21}, a_{31}, a_{12}, a_{22}, a_{23}.
- They can be inferred by giving the correspondence of three 2-D points between the input and output images. That is,

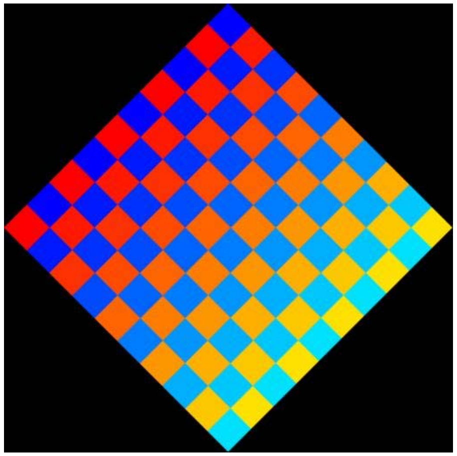
$$\left.\begin{array}{lcl}(u_1,v_1) & \rightarrow & (x_1,y_1)\\(u_2,v_2) & \rightarrow & (x_2,y_2)\\(u_3,v_3) & \rightarrow & (x_3,y_3)\end{array}\right\}\ 6\,\text{constraints}: (3\,\text{for}\ u \rightarrow x,\ 3\,\text{for}\ v \rightarrow y)$$

- All points lie on the same plane.
- Affine transformations map triangles onto triangles.

Affine Transformations (3)


Skew (shear)


Rotation/Scale


Rotation

Perspective Transformations (1)

$$[x', y', w'] = [u, v, w] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

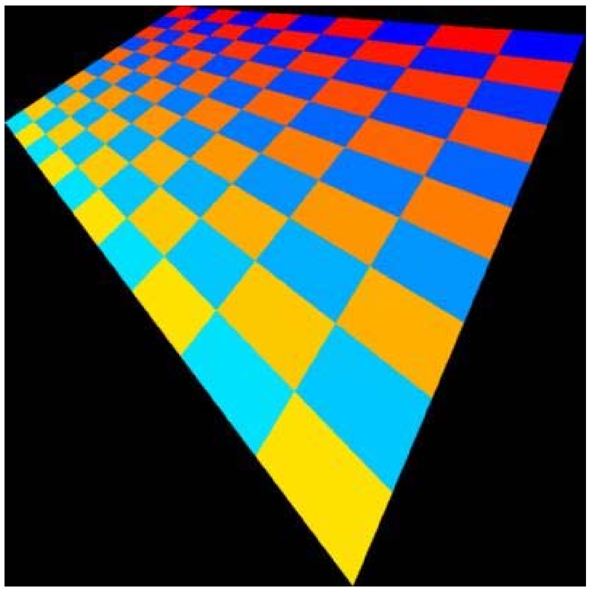
$$x = \frac{x'}{w'} = \frac{a_{11}u + a_{21}v + a_{31}}{a_{13}u + a_{23}v + a_{33}}$$

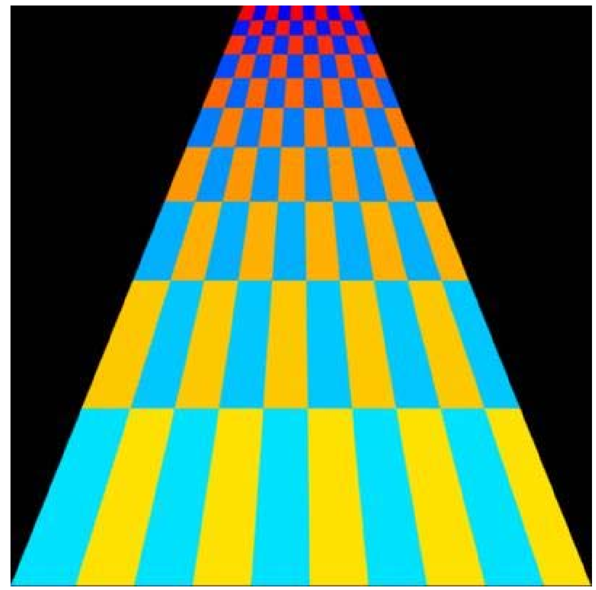
$$y = \frac{y'}{w'} = \frac{a_{12}u + a_{22}v + a_{32}}{a_{13}u + a_{23}v + a_{33}}$$

$$\begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \text{not necessarily} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{as in affine transformations}$$

- Without loss of generality, we set a_{33}=1.
- This yields 8 degrees of freedom and allows us to map planar quadrilaterals to planar quadrilaterals (correspondence among 4 sets of 2-D points yields 8 coordinates).
- Perspective transformations introduces foreshortening effects.
- Straight lines are preserved.

Perspective Transformations (2)



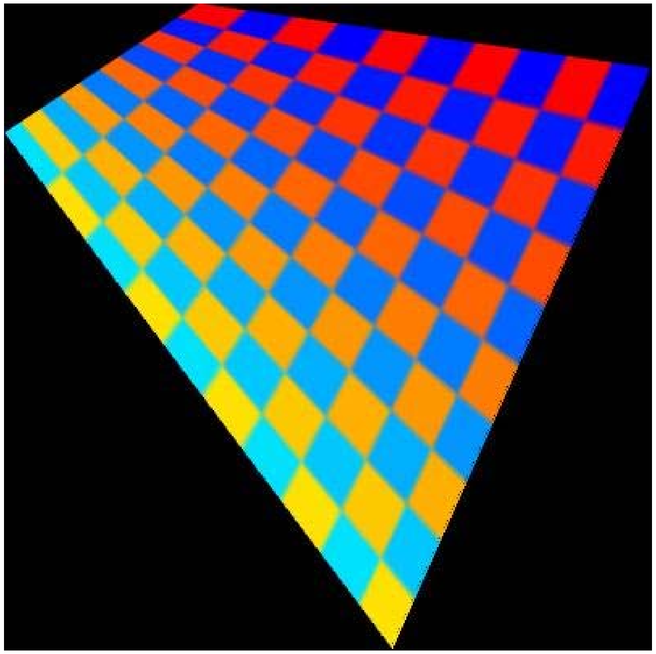


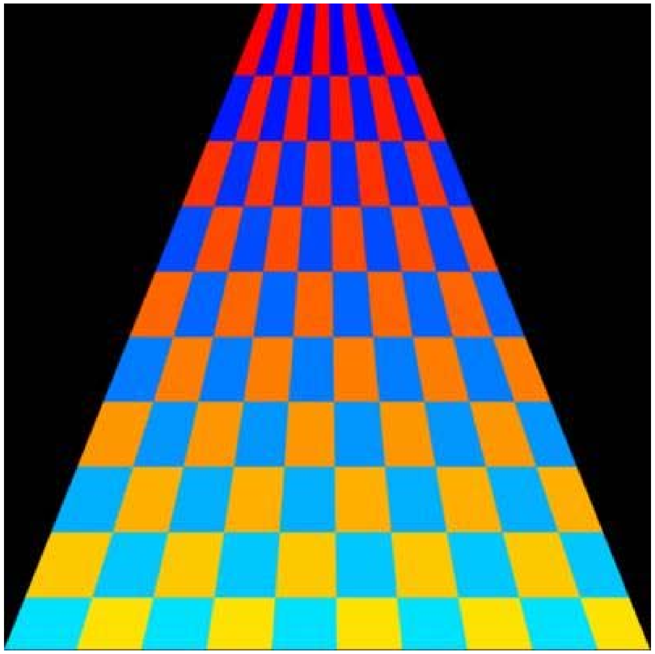
Bilinear Transforms (1)

$$[x, y] = [uv, u, v, 1]\begin{bmatrix} a_3 & b_3 \\ a_2 & b_2 \\ a_1 & b_1 \\ a_0 & b_0 \end{bmatrix}$$

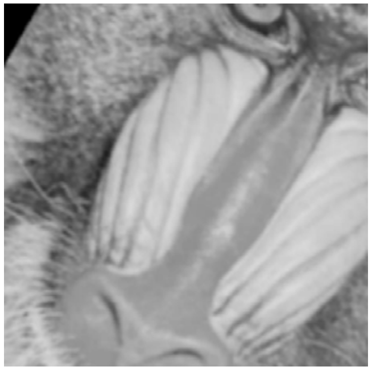
- 4-corner mapping among nonplanar quadrilaterals (2nd degree due to *uv* factors).
- Conveniently computed using separability (see p. 57-60).
- Preserves spacing along edges.
- Straight lines in the interior no longer remain straight.

Bilinear Transforms (2)

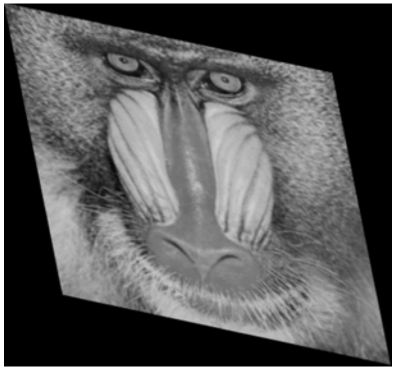




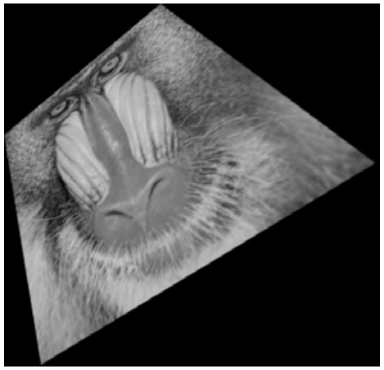
Examples



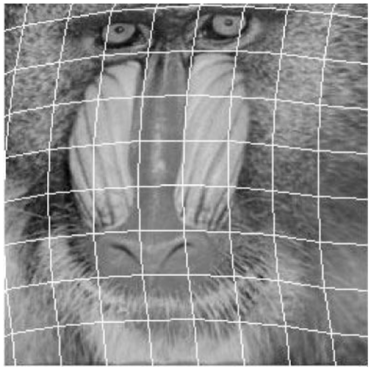
Similarity transformation (RST)



Affine transformation



Perspective transformation



Polynomial transformation

Scanline Algorithms

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Objectives

- In this lecture we review scanline algorithms:
  - Incremental texture mapping
  - 2-pass Catmull-Smith algorithm
    - Rotation
    - Perspective
  - 3-pass shear transformation for rotation
  - Morphing

Catmull-Smith Algorithm

- Two-pass transform
- First pass resamples all rows: $[u, v] \rightarrow [x, v]$

  $[x, v] = [F_v(u), v]$ where $F_v(u) = X(u, v)$ is the forward mapping fct
- Second pass resamples all columns: $[x, v] \rightarrow [x, y]$

  $[x, y] = [x, G_x(v)]$ where $G_x(v)=Y(H_x(v), v)$
- $H_x(v)$ is the inverse projection of x', the column we wish to resample.
- It brings us back from *[x, v]* to *[u, v]* so that we can directly index into Y to get the destination y coordinates.

Example: Rotation (1)

$$[x, y] = [u, v]\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Pass 1: $[x, v] = [u\cos\theta - v\sin\theta, v]$

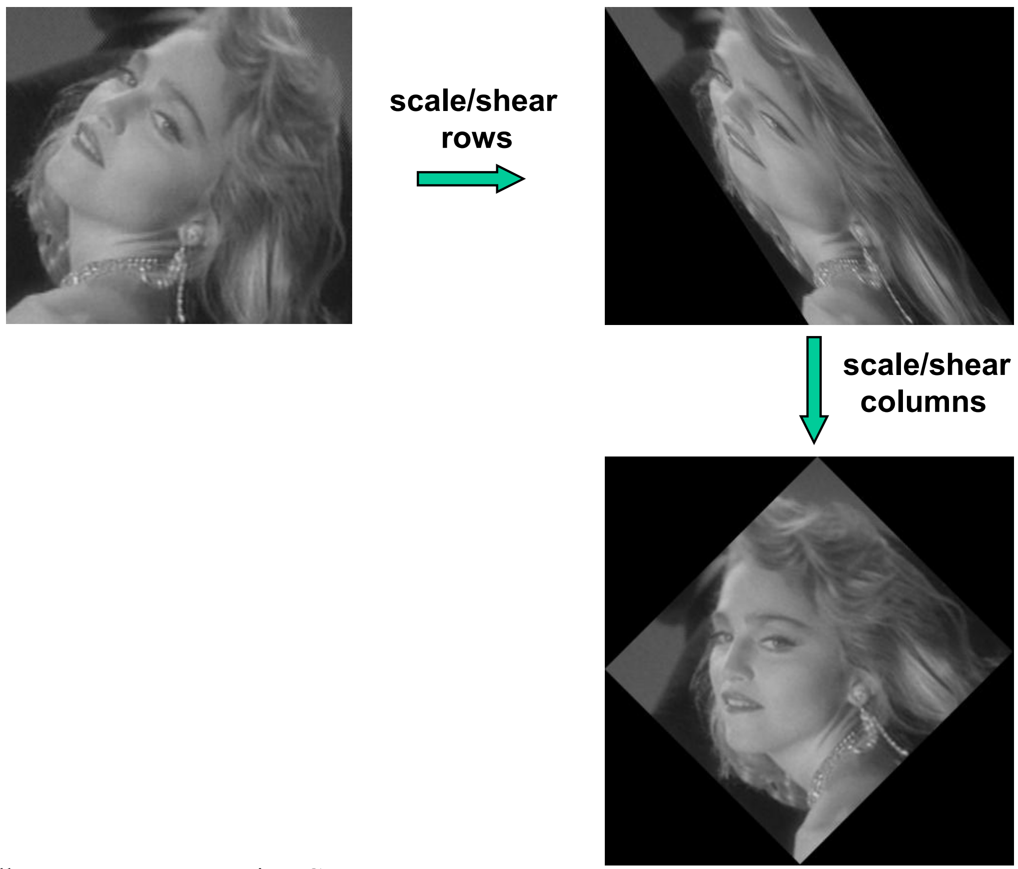
Pass 2: a) Compute $H_x(v)$. Recall that $x = u\cos\theta - v\sin\theta$

$$u = \frac{x + v\sin\theta}{\cos\theta}$$

b) Compute $G_x(v)$. Substitute $H_x(v)$ into $y = u\sin\theta + v\cos\theta$

$$y = \frac{x\sin\theta + v}{\cos\theta}$$

2-Pass Rotation



3-Pass Rotation Algorithm

- Rotation can be decomposed into two scale/shear matrices.

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ -\sin\theta & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\theta \\ 0 & \dfrac{1}{\cos\theta} \end{bmatrix}$$

- Three pass transform uses on shear matrices.

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\tan\left(\dfrac{\theta}{2}\right) & 1 \end{bmatrix} \begin{bmatrix} 1 & \sin\theta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\tan\left(\dfrac{\theta}{2}\right) & 1 \end{bmatrix}$$

- Advantage of 3-pass: no scaling necessary in any pass.

3-Pass Rotation

Software Implementation

- The following slides contain code for **`initMatrix.c`** to produce a 3x3 perspective transformation matrix from a list of four corresponding points (e.g. image corners).
- That matrix is then used in **`perspective.c`** to resample the image.
- The code in **`resample.c`** performs the actual scanline resample.

initMatrix.c (1)

```
/* ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
 * initMatrix:
 *
 * Given Icorr, a list of the 4 correspondence points for the corners
 * of image I, compute the 3x3 perspective matrix in Imatrix.
 */
void initMatrix(imageP I, imageP Icorr, imageP Imatrix)
{
  int     w,  h;
  float  *p, *a, a13, a23;
  float  x0, x1, x2, x3;
  float  y0, y1, y2, y3;
  float  dx1, dx2, dx3, dy1, dy2, dy3;

  /* init pointers */
  a = (float *) Imatrix->buf[0];
  p = (float *)   Icorr->buf[0];

  /* init u,v,x,y vars and print them */
  x0 = *p++;     y0 = *p++;
  x1 = *p++;     y1 = *p++;
  x2 = *p++;     y2 = *p++;
  x3 = *p++;     y3 = *p++;
```

initMatrix.c (2)

```
w = I->width;
h = I->height;
UI_printf("\nCorrespondence points:\n");
UI_printf("%4d %4d %6.1f %6.1f\n", 0, 0, x0, y0);
UI_printf("%4d %4d %6.1f %6.1f\n", w, 0, x1, y1);
UI_printf("%4d %4d %6.1f %6.1f\n", w, h, x2, y2);
UI_printf("%4d %4d %6.1f %6.1f\n", 0, h, x3, y3);

/* compute auxiliary vars */
dx1 = x1 - x2;
dx2 = x3 - x2;
dx3 = x0 - x1 + x2 - x3;
dy1 = y1 - y2;
dy2 = y3 - y2;
dy3 = y0 - y1 + y2 - y3;
```

initMatrix.c (3)

```
  /* compute 3x3 transformation matrix:
   * a0 a1 a2
   * a3 a4 a5
   * a6 a7 a8
   */
  a13  = (dx3*dy2 - dx2*dy3) / (dx1*dy2 - dx2*dy1);
  a23  = (dx1*dy3 - dx3*dy1) / (dx1*dy2 - dx2*dy1);
  a[0] = (x1-x0+a13*x1) / w;
  a[1] = (y1-y0+a13*y1) / w;
  a[2] = a13 / w;
  a[3] = (x3-x0+a23*x3) / h;
  a[4] = (y3-y0+a23*y3) / h;
  a[5] = a23 / h;
  a[6] = x0;
  a[7] = y0;
  a[8] = 1;
}
```

perspective.c (1)

```
#define X(A, U, V)       ((A[0]*U + A[3]*V + A[6]) / (A[2]*U + A[5]*V + A[8]))
#define Y(A, U, V)       ((A[1]*U + A[4]*V + A[7]) / (A[2]*U + A[5]*V + A[8]))
#define H(A, X, V)       ((-(A[5]*V+A[8])*X+ A[3]*V + A[6]) / (A[2]*X - A[0]))


/* ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
 * perspective:
 *
 * Apply a perspective image transformation on input I1.
 * The 3x3 perspective matrix is given in Imatrix.
 * The output is stored in I2.
 */
void perspective(imagP I1, imageP Imatrix, imageP I2)
{
  int    i, w, h, ww, hh;
  uchar  *p1, *p2;
  float  u, v, x, y, xmin, xmax, ymin, ymax, *a, *F;
  imageP II;

  w = I1->width;
  h = I1->height;
  a = (float *) Imatrix->buf[0];
```

perspective.c (2)

```
xmin = xmax = X(a, 0, 0);
x = X(a, w, 0); xmin = MIN(xmin, x);      xmax = MAX(xmax, x);
x = X(a, w, h); xmin = MIN(xmin, x);      xmax = MAX(xmax, x);
x = X(a, 0, h); xmin = MIN(xmin, x);      xmax = MAX(xmax, x);

ymin = ymax = Y(a, 0, 0);
y = Y(a, w, 0); ymin = MIN(ymin, y);      ymax = MAX(ymax, y);
y = Y(a, w, h); ymin = MIN(ymin, y);      ymax = MAX(ymax, y);
y = Y(a, 0, h); ymin = MIN(ymin, y);      ymax = MAX(ymax, y);

ww = CEILING(xmax) - FLOOR(xmin);
hh = CEILING(ymax) - FLOOR(ymin);

/* allocate mapping fct buffer */
x =  MAX(MAX(w, h), MAX(ww, hh));
F = (float *) malloc(x * sizeof(float));
if(F == NULL) IP_bailout("perspective: No memory");

/* allocate intermediate image */
II = IP_allocImage(ww, h, BW_TYPE);
IP_clearImage(II);
p1 = (uchar *) I1->buf[0];
p2 = (uchar *) II->buf[0];
```

perspective.c (3)

```
/* first pass: resample rows */
for(v=0; v<h; v++) {
      /* init forward mapping function F; map xmin to 0 */
      for(u=0; u< w; u++) F[(int) u] = X(a, u, v) - xmin;

      resample(p1, w, 1, F, p2);
      p1 += w;
      p2 += ww;
}

/* display intermediate image */
IP_copyImage(II, NextImageP);
IP_displayImage();

/* init final image */
IP_copyImageHeader(I1, I2);
I2->width  = ww;
I2->height = hh;
IP_initChannels(I2, BW_TYPE);
IP_clearImage(I2);
```

perspective.c (4)

```
/* second pass: resample columns */
for(x=0; x<ww; x++) {
        p1 = (uchar *) II->buf[0] + (int) x;
        p2 = (uchar *) I2->buf[0] + (int) x;

        /* skip past padding */
        for(v=0; v<h; v++,p1+=ww) {
                if(*p1) break;              /* check for nonzero pixel */
                u = H(a, (x+xmin), v);      /* else, if pixel is black */
                if(u>=0 && u<w) break;      /* then check for valid u  */
        }

        /* init forward mapping function F; map ymin to 0 */
        for(i=0; v<h; v++) {
                u = H(a, (x+xmin), v);
                u = CLIP(u, 0, w-1);
                F[i++] = Y(a, u, v) - ymin;
        }
        resample(p1, i, ww, F, p2);
}
IP_freeImage(II);
}
```

resample.c (1)

```
/* ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
 * Resample the len elements of src (with stride offst) into dst according
 * to the monotonic spatial mapping given in F (len entries).
 * The scanline is assumed to have been cleared earlier.
 */
void resample(uchar *src, int len, int offst, float *F, uchar *dst)
{
  int     u, uu, x, xx, ix0, ix1, I0, I1, pos;
  double  x0, x1, dI;

  if(F[0] < F[len-1])
        pos = 1;         /* positive output stride */
  else  pos = 0;         /* negative output stride */

  for(u=0; u<len-1; u++) {
        /* index into src */
        uu = u * offst;

        /* output interval (real and int) for input pixel u */
        if(pos) {/* positive stride */
                ix0 = x0 = F[u];
                ix1 = x1 = F[u+1];
                I0  = src[uu];
                I1  = src[uu+offst];
        }
```

resample.c (2)

```
else {  /* flip interval to enforce positive stride */
        ix0 = x0 = F[u+1];
        ix1 = x1 = F[u];
        I0  = src[uu+offst];
        I1  = src[uu];
}

/* index into dst */
xx = ix0 * offst;

/* check if interval is embedded in one output pixel */
if(ix0 == ix1) {
        dst[xx] += I0 * (x1-x0);
        continue;
}
/* else, input straddles more than one output pixel  */

/* left straddle */
dst[xx] += I0 * (ix0+1-x0);
```

resample.c (3)

```
        /* central interval */
        xx += offst;
        dI  = (I1-I0) / (x1-x0);
        for(x=ix0+1; x<ix1; x++,xx+=offst)
                dst[xx] = I0 + dI*(x-x0);

        /* right straddle */
        if(x1 != ix1)
                dst[xx] += (I0 + dI*(ix1-x0)) * (x1-ix1);
  }
}
```